The fiftieth

Seminar

Sophus Lie,

Bedlewo, XXV.IX. - I.X.MMXVI

Lie Calculus, Groupoids, and Loops

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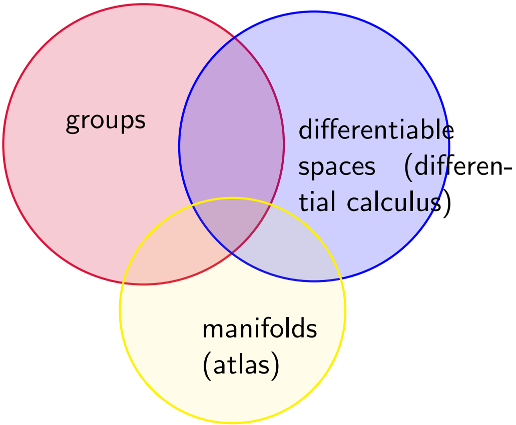

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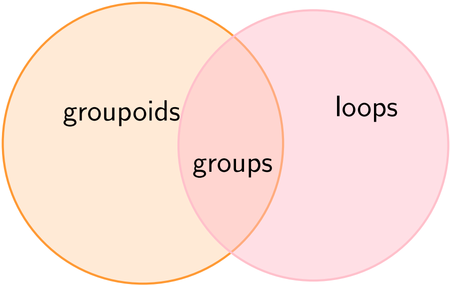

Definition. A *quasigroup* $(Q, \cdot)$ is a set Q together with a binary product $\cdot : Q \times Q \to Q$, $(a, b) \mapsto a \cdot b$, such that, for each $y \in Q$, the left translations $x \mapsto y \cdot x$ and the right translations $x \mapsto x \cdot y$ are bijective maps from Q to Q.

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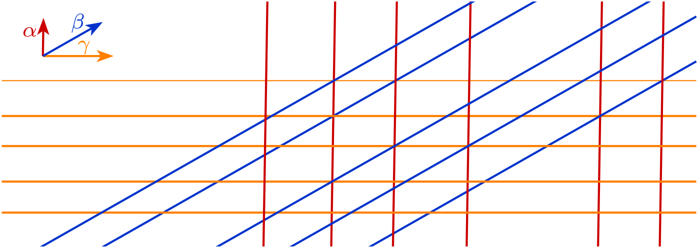

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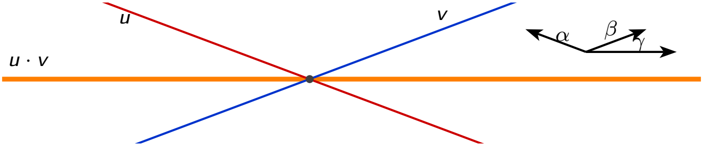

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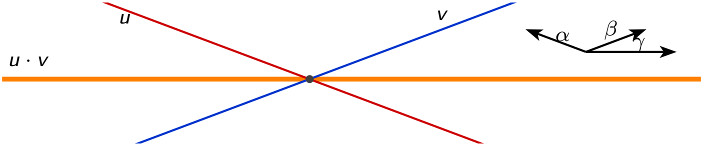

Theorem (easy: cf. Postmodern Algebra). *The product* $A \times B \to C$ *is a* dissociated (three-based) quasigroup*, i.e., left and right translations are bijections. And so are the other five products, called* parastrophic *with the first one.*

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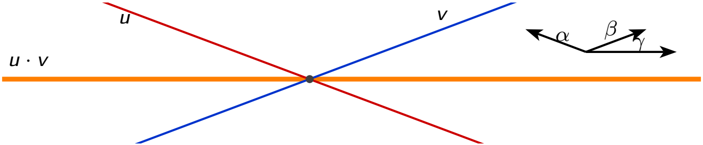

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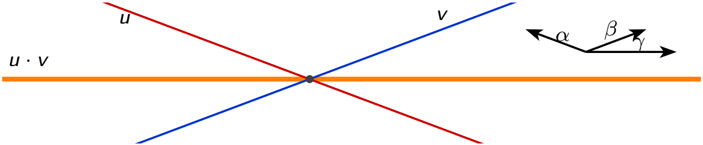

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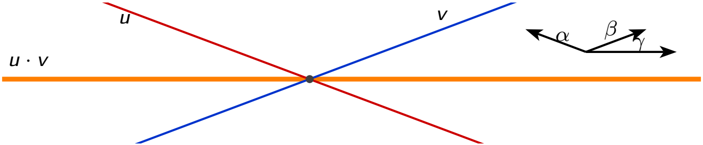

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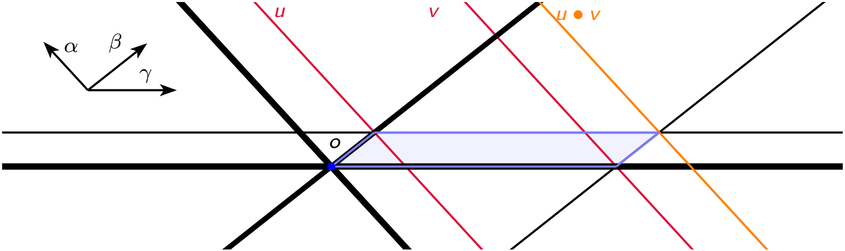

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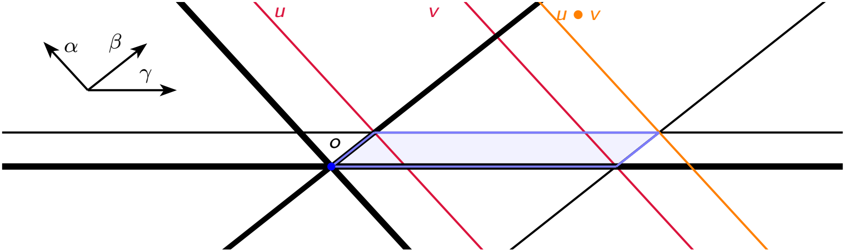

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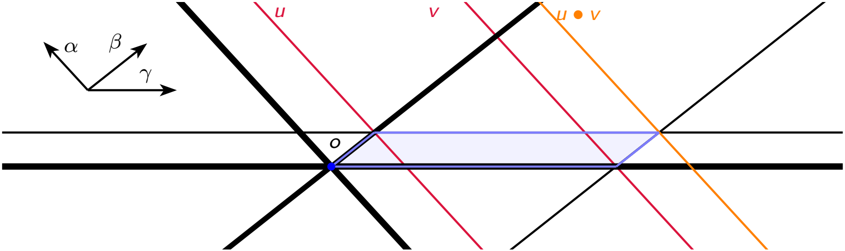

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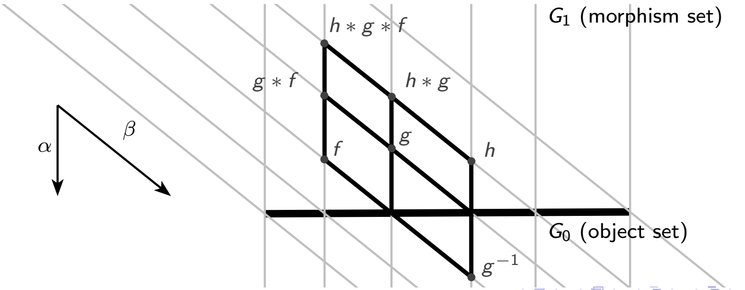
A groupoid may be visualized like this:



Groupoids: examples

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Open problems. What can we do if $\mathbb{K}$ is discrete (e.g., finite)? And what about (non) commutativity of $\mathbb{K}$?

The Chain Rule, and "Conceptual Calculus"

Notation: $\mathbb{K}$ good (as above), $f : V \supset U \to W$ as above.

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Proof. For $t \in \mathbb{K}^\times$, direct computation; for $t = 0$ by continuous extension.

"Conceptual calculus": study this functor! Its main feature is related to the fact that the differential should be a linear map:

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Methodological remark. In usual calculus, linearity of the differential is imposed *by definition*. In BGN-calculus, it is a *theorem*. By *Occam's razor*, this is an argument in favor of BGN-calculus.

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Theorem. *Gluing data with this equivalence relation and this partial order define an ordered groupoid.*

Thus manifold data form another instance of groupoids ([B, arxiv, 2016]).

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Forgetting the manifold (atlas), we still have a double groupoid.

Example. $G = \mathrm{GL}(n, \mathbb{K})$: a single chart (the natural one) suffices, so the atlas is the trivial groupoid. The set $G^{\{1\}}$ has two groupoid structures, one of which is a group, but the other not:

$$\begin{array}{ccc} G^{\{1\}} & \rightarrow & \{e\}^{\{1\}} = \mathbb{K} \\ \downarrow\downarrow & & \downarrow \\ G \times \mathbb{K} & \rightarrow & \{e\} \times \mathbb{K} \end{array}$$

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as well as a diagram of unit sections, and products $*$ (on C_{11} and C_{10}) and $\bullet$ (on C_{11} and C_{01}), and inversions, such that:

- *each edge of the diagram forms a groupoid,*
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Example. If all edge groupoids are *groups*, then the interchange law forces $*$ and $\bullet$ to be the same, *commutative* group law.

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n-fold groupoids

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Theorem. (Folklore among category theorists? [B, arxiv 2015]) *Equivalently, an n-fold groupoid is given by 2^n sets C_i, $i \in I$, where the index set I is an n-hypercube, with diagrams of source and target projections, unit sections, and with products and inversions such that*

- *each edge diagram represents a groupoid,*
- *each face diagram represents a double groupoid.*

Thus n-fold groupoids are "tamed": they are algebraic structures in the usual sense.

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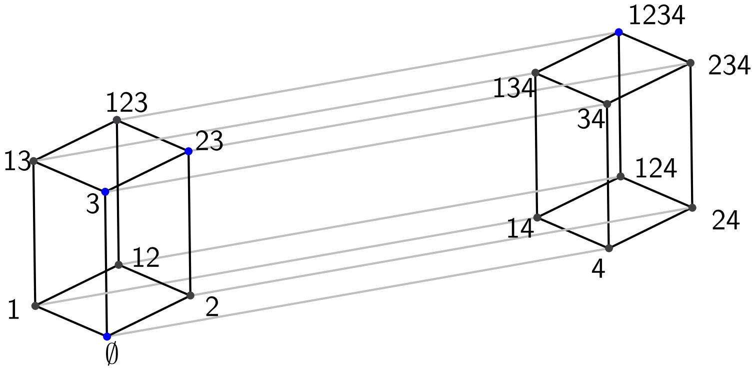

This figure illustrates the standard induction step from 3 to 4.

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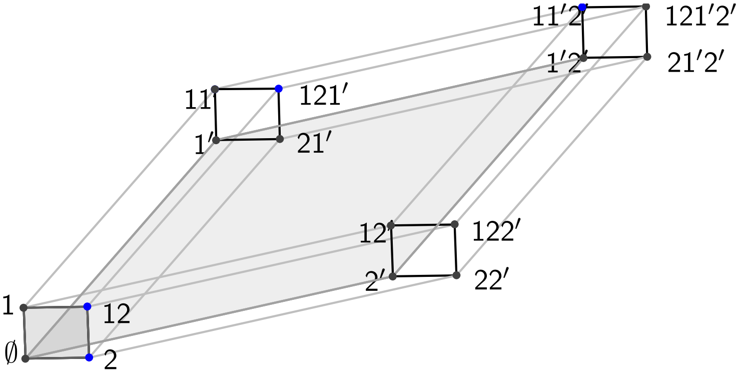

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[Note : elements (x, v, t) of $U^{\{1\}}$ are triples, elements of $U^{\{1,2\}}$ are 7-tuples, and elements of $U^{\{1,\ldots,n\}}$ are $2^{n+1} - 1$-tuples. This looks bad!]

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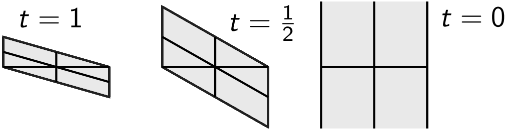

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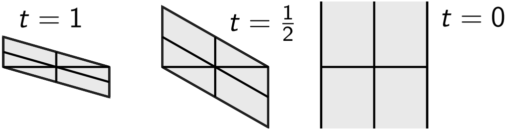

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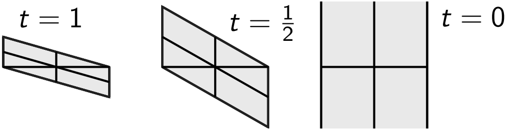

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Summary: understanding the higher order theory of Lie groups (or groupoids) and understanding higher order calculus is equivalent. That's why I call it "Lie Calculus".

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- is there a "super-scaleoid" ?

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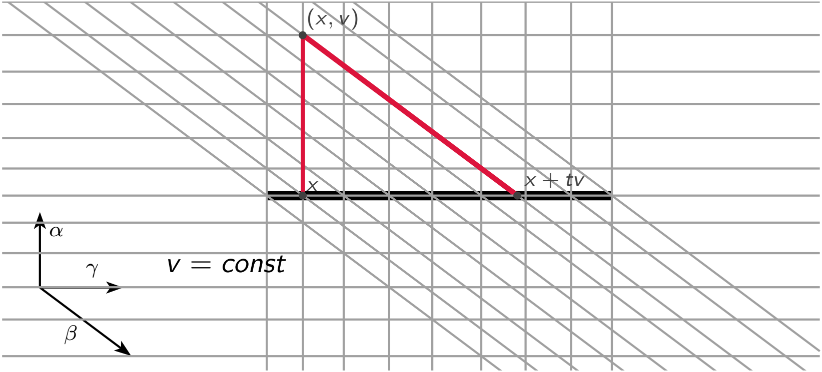
Well, here they are: they sit in the groupoid $U^{\{1\}}$:



The horizontal distribution γ given by $v = const$ represents the *canonical flat connection of the ambiant vector space V*. This defines a 3-web, hence a loop!

Loops, connections, and differential geometry

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Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.' (Nowadays you can think of it as a computer!) Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.