A Lie systems approach to the Riccati hierarchy and PDEs

Javier de Lucas

(Joint work with A.M. Grundland)

Department of Mathematical Methods in Physics
University of Warsaw

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Superposition rules and Lie systems

A *Riccati equation* is a differential equation on $\mathbb{R}$ of the form

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2, \qquad b_1(t), b_2(t), b_3(t) \text{ arbitrary.}$$

Its general solution, $x(t)$, can be written as

$$x(t) = \frac{x_{(1)}(t)(x_{(3)}(t) - x_{(2)}(t)) - k\, x_{(2)}(t)(x_{(3)}(t) - x_{(1)}(t))}{(x_{(3)}(t) - x_{(2)}(t)) - k\,(x_{(3)}(t) - x_{(1)}(t))},$$

in terms of different particular solutions $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ and $k \in \mathbb{R}$. So, we can write

$$x(t) = \Phi(x_{(1)}(t), x_{(2)}(t), x_{(3)}(t); k)$$

for $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi(x_{(1)}, x_{(2)}, x_{(3)}; k) := \frac{x_{(1)}(x_{(3)} - x_{(2)}) - k\, x_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k\,(x_{(3)} - x_{(1)})}.$$

The function Φ is an example of what is referred to as *superposition rule*.

Definition

A *superposition rule* for a system of first-order ordinary differential equations on N is a function $\Phi : N^m \times N \to N$, $x = \Phi(x_{(1)}, \ldots, x_{(m)}; k)$, such that the general solution $x(t)$ of our system can be brought into the form

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k),$$

where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is any generic family of particular solutions and k is a point of N to be related to initial conditions.

Definition

A *Lie system* is a system of first-order ordinary differential equations admitting a *superposition rule*.

Example

So Riccati equations are Lie systems with a superposition rule

$$\Phi : \mathbb{R}^3 \times \mathbb{R} \ni (x_{(1)}, x_{(2)}, x_{(3)}; k) \mapsto \frac{x_{(1)}(x_{(3)} - x_{(2)}) - k\, x_{(2)}(x_{(3)} - x_{(1)})}{(x_{(3)} - x_{(2)}) - k\,(x_{(3)} - x_{(1)})} \in \mathbb{R}.$$

Characterisation of Lie systems

We can associate every Riccati equation with a t-dependent vector field

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)x^2 \Rightarrow X(t,x) := (b_1(t) + b_2(t)x + b_3(t)x^2)\frac{\partial}{\partial x},$$

admitting the decomposition

$$X(t,x) = b_1(t)X_1(x) + b_2(t)X_2(x) + b_3(t)X_3(x),$$

where

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = x^2\frac{\partial}{\partial x},$$

satisfy the commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3.$$

That is, X_1, X_2, X_3 span a Lie algebra of vector fields $V_{\rm Ric} \simeq \mathfrak{sl}(2,\mathbb{R})$.

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Proposition (Lie 1880)

Every finite-dimensional Lie algebra of vector fields on the real line is locally diffeomorphic, at a generic point, to a Lie subalgebra of $V_{\rm Ric}$.

Every first-order ordinary system of differential equations in normal form amounts to a t-dependent vector field.

$$X(t,x) = \sum_{i=1}^{n} X^i(t,x)\frac{\partial}{\partial x^i} \Longleftrightarrow \frac{dx^i}{dt} = X^i(t,x), \quad i = 1,\ldots,n.$$

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Lie–Scheffers Theorem (1893)

A system X on N admits a superposition rule depending on m particular solutions if and only

$$X=\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha$$

for a family of vector fields $X_1,\ldots,X_r$ on N spanning a finite-dimensional Lie algebra V of vector fields and certain t-dependent functions $b_1(t),\ldots,b_r(t)$. We call V a *Vessiot–Guldberg Lie algebra* for the Lie system. Additionally, it holds that $mn\geq r$.

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Proposition

All Lie systems on the real line are locally diffeomorphic to a Riccati equation.

Vessiot–Guldberg Lie algebras on $\mathbb{R}^2$

Primitive Lie algebras

Every Vessiot–Guldberg Lie algebra on $\mathbb{R}^2$ is locally diffeomorphic around a generic point to:

#	Primitive	Basis of vector fields X_i
P_1	$A_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, \alpha(x\partial_x + y\partial_y) + y\partial_x - x\partial_y, \quad \alpha \geq 0$
P_2	$\mathfrak{sl}(2)$	$\partial_x, x\partial_x + y\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y$
P_3	$\mathfrak{so}(3)$	$y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (1 + y^2 - x^2)\partial_y$
P_4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y$
P_5	$\mathfrak{sl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y$
P_6	$\mathfrak{gl}(2) \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y$
P_7	$\mathfrak{so}(3,1)$	$\partial_x, \partial_y, x\partial_x + y\partial_y, y\partial_x - x\partial_y, (x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (y^2 - x^2)\partial_y$
P_8	$\mathfrak{sl}(3)$	$\partial_x, \partial_y, x\partial_x, y\partial_x, x\partial_y, y\partial_y, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y$

#	Imprimitive	Basis of vector fields X_i $(r \geq 1)$
I_2	$\mathfrak{h}_2$	$\partial_x, x\partial_x$
I_3	$\mathfrak{sl}(2)$ (type I)	$\partial_x, x\partial_x, x^2\partial_x$
I_4	$\mathfrak{sl}(2)$ (type II)	$\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y$
I_5	$\mathfrak{sl}(2)$ (type III)	$\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y$
I_6	$\mathfrak{gl}(2)$ (type I)	$\partial_x, \partial_y, x\partial_x, x^2\partial_x$
I_7	$\mathfrak{gl}(2)$ (type II)	$\partial_x, y\partial_y, x\partial_x, x^2\partial_x + xy\partial_y$
I_8	$B_\alpha \simeq \mathbb{R} \ltimes \mathbb{R}^2$	$\partial_x, \partial_y, x\partial_x + \alpha y\partial_y, \quad 0 < \lvert\alpha\rvert \leq 1$
I_9	$\mathfrak{h}_2 \oplus \mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y$
I_{10}	$\mathfrak{sl}(2) \oplus \mathfrak{h}_2$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x$
I_{11}	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$	$\partial_x, \partial_y, x\partial_x, y\partial_y, x^2\partial_x, y^2\partial_y$
I_{12}	$\mathbb{R}^{r+1}$	$\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y$
I_{13}	$\mathbb{R} \ltimes \mathbb{R}^{r+1}$	$\partial_y, y\partial_y, \xi_1(x)\partial_y, \ldots, \xi_r(x)\partial_y$
I_{14}	$\mathbb{R} \ltimes \mathbb{R}^r$	$\partial_x, \eta_1(x)\partial_y, \eta_2(x)\partial_y, \ldots, \eta_r(x)\partial_y$
I_{15}	$\mathbb{R}^2 \ltimes \mathbb{R}^r$	$\partial_x, y\partial_y, \eta_1(x)\partial_y, \ldots, \eta_r(x)\partial_y$
I_{16}	$C^r_\alpha \simeq \mathfrak{h}_2 \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x + \alpha y\partial y, x\partial_y, \ldots, x^r\partial_y, \qquad \alpha \in \mathbb{R}$
I_{17}	$\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^r)$	$\partial_x, \partial_y, x\partial_x + (ry + x^r)\partial_y, x\partial_y, \ldots, x^{r-1}\partial_y$
I_{18}	$(\mathfrak{h}_2 \oplus \mathbb{R}) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_y, \ldots, x^r\partial_y$
I_{19}	$\mathfrak{sl}(2) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_y, 2x\partial_x + ry\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y$
I_{20}	$\mathfrak{gl}(2) \ltimes \mathbb{R}^{r+1}$	$\partial_x, \partial_y, x\partial_x, x\partial_y, y\partial_y, x^2\partial_x + rxy\partial_y, x^2\partial_y, \ldots, x^r\partial_y$

Projective Riccati equations

Definition

Projective Riccati equations take the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = b_0(x) + C(x)\xi + \langle \xi, b_2(x) \rangle \xi, \qquad \xi \in \mathbb{R}^n, \tag{1}$$

where $C(x)$ is a real $n \times n$ matrix, and $b_0(x), b_2(x) \in \mathbb{R}^n$ for every $x \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ is the standard Euclidean metric.

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Proposition (Anders-Harnad-Winternitz 1980)

Each projective Riccati equation on $\mathbb{R}^n$ is a Lie system associated with a Vessiot–Guldberg Lie algebra $V_n^{\mathrm{Pr}} \simeq \mathfrak{sl}(n+1, \mathbb{R})$.

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A *projective vector field* Z on a pseudo-Riemannian manifold (N, g) is a vector field Z on N whose flow maps geodesics of the metric g into new geodesics.

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Projective vector fields on $\mathbb{R}^n$ relative to $\langle \cdot, \cdot \rangle$ are $\langle \partial_i, \xi^j \partial_i, \xi^j \sum_{k=1}^n \xi^k \partial_k \rangle$ with $i, j = \overline{1, n}$. They span a Lie algebra isomorphic to $\mathfrak{sl}(n+1, \mathbb{R})$.

Superposition rules for the projective Riccati equations

(Proposition Anders-Harnad-Winternitz 1980/1981)

A projective Riccati equation on $\mathbb{R}^n$ admits a superposition rule in terms of $n+2$ generic particular solutions of the form

$$\Psi : \mathbb{R}^{n(n+2)} \times \mathbb{R}^n \ni (\xi_{(1)}, \ldots, \xi_{(n+2)}; \chi) \mapsto \xi := \frac{B\chi + \rho}{\langle \sigma, \chi \rangle + b} \in \mathbb{R}^n,$$

where B is an $n \times n$ matrix with entries $B^\mu_k := \xi^\mu_{(k)}\sigma_k$, $\xi_{(k)} := (\xi^1_{(k)}, \ldots, \xi^n_{(k)})^T$,

$$\sigma_k := \det(\xi_{(1)} - \xi_{(n+1)}, \ldots, \overbrace{\xi_{(n+2)} - \xi_{(n+1)}}^{k-\mathrm{term}}, \xi_{(n)} - \xi_{(n+1)}), \ \ k = 1, \ldots, n,$$

$B\chi$ is the matrix multiplication of B with $\chi := (\chi_1, \ldots, \chi_n)^T$, $\sigma := (\sigma_1, \ldots, \sigma_n)^T$ and

$$b := \left(1 - \sum_{k=1}^n \chi_k\right) \det(\xi_{(1)} - \xi_{(n+1)}, \ldots, \xi_{(n)} - \xi_{(n+1)}), \qquad \rho := b\,\xi_{(n+2)},$$

for $k, \mu = 1, \ldots, n$.

Conformal Riccati equations

Definition

A conformal Riccati equation takes the form

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = b_0(x) + A(x)\xi + \gamma(x)\xi + b_2(x)\langle\xi,\xi\rangle_{p,q} - 2\langle\xi, b_2(x)\rangle\xi, \qquad \xi \in \mathbb{R}^n,$$

where $\gamma(x)$ is an x-dependent real function, $\langle A(x)\xi_1, \xi_2\rangle_{p,q} + \langle\xi_1, A(x)\xi_2\rangle_{p,q} = 0$ for every $\xi_1, \xi_2 \in \mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ is a metric of signature (p, q) with $p + q = n$.

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Proposition (Anders-Harnad-Winternitz 1981)

Conformal Riccati equations are Lie systems related to a Vessiot–Guldberg Lie algebra $V^{(p,q)}$ of conformal vector fields relative to a flat metric of signature (p, q) and therefore isomorphic to $\mathfrak{so}(p+1, q+1)$.

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Conformal Riccati equations admit a superposition rule in terms of $n+1$ particular solutions.

Definition of the Riccati hierarchy

Definition

An *s-order Riccati chain equation* is a differential equation of the form

$$L_c^s u + \sum_{j=1}^{s} \alpha_j(x) L_c^{j-1} u + \alpha_0(x) = 0, \qquad u, x \in \mathbb{R}, \qquad c \in \mathbb{R} \backslash \{0\}, \qquad s \in \mathbb{N}, \tag{2}$$

where $\alpha_0(x), \ldots, \alpha_s(x)$ are arbitrary x-dependent real functions, $L^s := L \circ \cdots \circ L(s - \mathrm{times})$, $L_c^0 u := u$, and L_c is the differential operator on the real line given by

$$L_c := \frac{\mathrm{d}}{\mathrm{d}x} + c\, u, \qquad c \in \mathbb{R}. \tag{3}$$

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Examples: first- and second-order Riccati chain equations

$$\frac{\mathrm{d}u}{\mathrm{d}x}+cu^2+\alpha_1(x)u+\alpha_0(x)=0,$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+(\alpha_2(x)+3cu)\frac{\mathrm{d}u}{\mathrm{d}x}+c^2u^3+c\alpha_2(x)u^2+\alpha_1(x)u+\alpha_0(x)=0.$$

Riccati hierarchy and Lie systems

Theorem (Grundland-De Lucas 2016)

An s-order Riccati chain equation (2), when written as a first-order system on $\mathrm{T}^{s-1}\mathbb{R}$, can be mapped onto the projective Riccati equation on $\mathbb{R}^s$ with

$$b_0(x) := \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ -\alpha_0(x) \end{pmatrix}, \quad C(x) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_1(x) & -\alpha_2(x) & -\alpha_3(x) & \cdots & -\alpha_n(x) \end{pmatrix}, \quad b_2(x) := \begin{pmatrix} -c \\ 0 \\ \cdots \\ 0 \\ 0 \end{pmatrix},$$

via a global diffeomorphism
$\phi_{s,c} : (u^{0)}, \ldots, u^{s-1)}) \in \mathrm{T}^{s-1}\mathbb{R} \mapsto (y_1, \ldots, y_s)^T \in \mathbb{R}^s$, with $\mathrm{T}^0\mathbb{R} := \mathbb{R}$ and

$$y_k(x) := L_c^{k-1} u(x), \qquad k = 1, \ldots, s. \tag{4}$$

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Proposition

An s-order Riccati chain equation in first-order form admits a Vessiot–Guldberg Lie algebra V_s^{RC} of projective vector fields for the flat Riemannian metric

$$g_s^{\mathrm{RC}} := \sum_{i=0}^{s-1} \mathrm{d}(L^i u) \otimes \mathrm{d}(L^i u). \tag{5}$$

Superposition rule for the Riccati hierarchi

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Proposition

Every s-order Riccati chain equation, when considered as a non-autonomous first-order system, admits a superposition rule depending on $s+2$ particular solutions of the form

$$\Psi_s : [\mathrm{T}^{s-1}\mathbb{R}]^{s+2} \times \mathbb{R}^s \longrightarrow \mathrm{T}^{s-1}\mathbb{R}$$

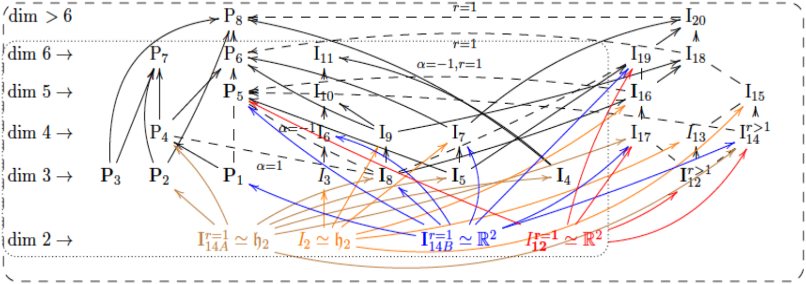
$$(\mathrm{t}^{s-1}u_{(1)}, \ldots, \mathrm{t}^{s-1}u_{(s+2)}, \chi) \longrightarrow \phi^{-1}_{s,c}\left(\frac{B\chi + b\phi_{s,c}(\mathrm{t}^{s-1}u_{(s+2)})}{\langle\sigma,\chi\rangle + b}\right)$$

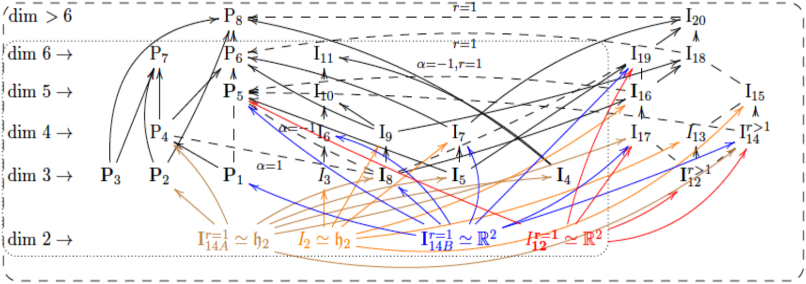
where $B^\mu_k := [\phi_{s,c}(\mathrm{t}^{s-1}u_{(k)})]^\mu \sigma_k (\text{no sum})$ and

$$b := \left(1 - \sum_{k=1}^{s} \chi_k\right) \det[\phi_{s,c}(\mathrm{t}^{s-1}u_{(1)}) - \phi_{s,c}(\mathrm{t}^{s-1}u_{(n+1)}), \ldots, \phi_{s,c}(\mathrm{t}^{s-1}u_{(n)}) - \phi_{s,c}(\mathrm{t}^{s-1}u_{(n+1)})],$$

$$\sigma_k := \det[\phi_{s,c}(\mathrm{t}^{s-1}u_{(1)}) - \phi_{s,c}(\mathrm{t}^{s-1}u_{(n+1)}), \ldots, \overbrace{\phi_{s,c}(\mathrm{t}^{s-1}u_{(n+2)}) - \phi_{s,c}(\mathrm{t}^{s-1}u_{(n+1)})}^{k-\text{term}}, \ldots, \phi_{s,c}(\mathrm{t}^{s-1}u_{(n)}) - \phi_{s,c}(\mathrm{t}^{s-1}u_{(n+1)})], \quad (6)$$

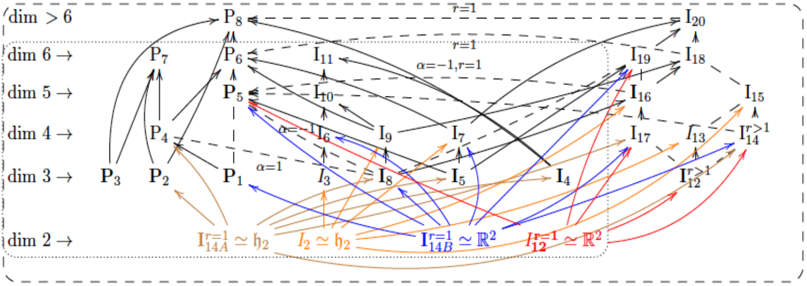
for $k, \mu = 1, \ldots, s$.


dim > 6
dim 6 →
dim 5 →
dim 4 →
dim 3 →
dim 2 →
P_8
P_7
P_6
P_5
P_4
P_3
P_2
P_1
$r=1$
$r=1$
$\alpha=-1, r=1$
$\alpha=-1$
$\alpha=1$
I_{20}
I_{19}
I_{18}
I_{11}
I_{10}
I_{16}
I_{15}
I_6
I_9
I_7
I_{17}
I_{13}
$I_{14}^{r>1}$
I_3
I_8
I_5
I_4
$I_{12}^{r>1}$
$I_{14A}^{r=1} \simeq \mathfrak{h}_2$
$I_2 \simeq \mathfrak{h}_2$
$I_{14B}^{r=1} \simeq \mathbb{R}^2$
$I_{12}^{r=1} \simeq \mathbb{R}^2$



Proposition

The classes $\mathrm{I}_1, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4, \mathrm{P}_7, \mathrm{I}_8^{\alpha=1}, \mathrm{I}_{14}^{r=1}$ are the only classes of Lie algebras of Euclidean vector fields on $\mathbb{R}^2$. They are the Lie subalgebras of P_7.



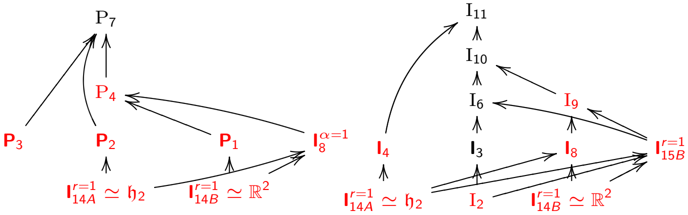
Proposition

The classes $\mathrm{I}_1, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{P}_4, \mathrm{P}_7, \mathrm{I}_8^{\alpha=1}, \mathrm{I}_{14}^{r=1}$ are the only classes of Lie algebras of Euclidean vector fields on $\mathbb{R}^2$. They are the Lie subalgebras of P_7.

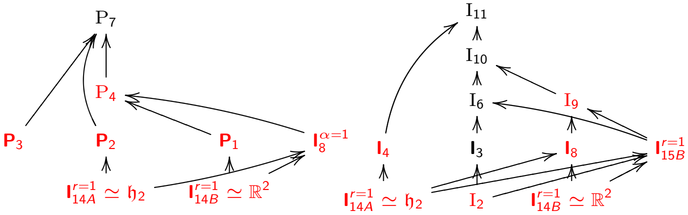
Proposition

A Vessiot–Guldberg Lie algebra on $\mathbb{R}^2$ consists of hyperbolic vector fields if and only if it is diffeomorphic to a Lie subalgebra of I_{11}, namely
$\mathrm{I}_1 - \mathrm{I}_4, \mathrm{I}_6, \mathrm{I}_8, \mathrm{I}_9 - \mathrm{I}_{11}, \mathrm{I}_{14B}^{r=1}, \mathrm{I}_{15B}^{r=1}$.

Lie algebras of Euclidean and hyperbolic vector fields on $\mathbb{R}^2$ and their inclusion relations. Lie algebras of projective vector fields are in red.



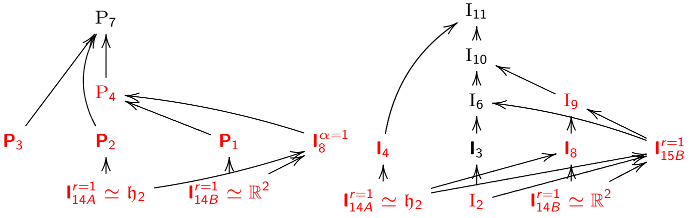
Lie algebras of Euclidean and hyperbolic vector fields on $\mathbb{R}^2$ and their inclusion relations. Lie algebras of projective vector fields are in red.



Proposition

Every Lie algebra of Euclidean and projective vector fields on $\mathbb{R}^2$ is diffeomorphic to a Lie subalgebra of P_2, P_3 or P_4.

Lie algebras of Euclidean and hyperbolic vector fields on $\mathbb{R}^2$ and their inclusion relations. Lie algebras of projective vector fields are in red.



Proposition

Every Lie algebra of Euclidean and projective vector fields on $\mathbb{R}^2$ is diffeomorphic to a Lie subalgebra of P_2, P_3 or P_4.

Proposition

A Vessiot–Guldberg Lie algebra of vector fields on $\mathbb{R}^2$ consists of projective hyperbolic vector fields if and only if it is diffeomorphic to a Lie subalgebra of I_4 or I_9.

Theorem

A not autonomous second-order Riccati chain equation can be mapped by an autonomous diffeomorphism into a Euclidean Riccati equation if and only if it takes the form:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = -3cu\frac{\mathrm{d}u}{\mathrm{d}x} - c^2u^3 + f(x)c_0 + c_0(c_1+\bar{c}_2) + \left[f(x)c_1 + c_1^2/2 - 1\right]u + [f(x)c_2+\bar{c}_2]\left(cu^2 + \frac{\mathrm{d}u}{\mathrm{d}x}\right). \tag{7}$$

for any non-constant x-dependent function $f(x)$, coefficients $c_1, c_0 \in \mathbb{R}$ such that $c_1^2 - 4c_0c_2c < 0$ with $c_2 \in \mathbb{R}\backslash\{0\}$, and any $\bar{c}_2 \in \mathbb{R}$. A not-autonomous second-order affine Riccati chain equation is diffeomorphic to a Euclidean Riccati equation if and only if

$$a)\ \alpha_1(x) = \alpha_2(x) = 0,\ \bar{c}_2^2 + 4\bar{c}_1 < 0 \qquad b)\ \alpha_0(x) = \alpha_1(x) = \bar{c}_0 = \bar{c}_1 = 0.$$

A not autonomous second-order Riccati chain equation is diffeomorphic, as a first-order system, to a hyperbolic Riccati equation when it takes the form (7) for $c_1^2 - 4c_0c_2c > 0$, $c, c_2 \neq 0$, a non-constant function $f(x)$, and arbitrary $\bar{c}_2 \in \mathbb{R}$. A not autonomous affine second-order Riccati chain equation is diffeomorphic to a hyperbolic Riccati equation if and only if it takes the form (7) for $c = 0$ and $c_2 \neq 0$ or

$$a)\ \alpha_1(x) = \alpha_2(x) = 0,\ \bar{c}_2^2 + 4\bar{c}_1 > 0 \qquad b)\ \alpha_0(x) = \alpha_1(x) = \bar{c}_0 = \bar{c}_1 = 0$$

$$c)\ c_1 = \bar{c}_1 = 0, c_1 \neq 0, c_0\bar{c}_2 - c_2\bar{c}_0 = 0.$$

Lie systems and Bäcklund transformations

Let u be a real function on $\mathbb{R}^2$. The Sawada–Kotera and Kaup–Kupershmidt equations take the form

$$u_t + (u_{4x} + 30uu_{xx} + 60u^3)_x = 0,$$
$$u_t + \left(u_{4x} + 30uu_{xx} + \frac{45}{2}u_x^2 + 60u^3\right)_x = 0,$$

respectively. Both partial differential equations are related to the linear spectral problem

$$\Psi_{xxx} + 6u\Psi_x + (6R - \lambda)\Psi = 0, \tag{8}$$

where $\Psi, R : \mathbb{R} \to \mathbb{R}$ are x-dependent functions and λ is a spectral parameter. More specifically, the linear spectral problem (8) for the SK equations has $R = 0$ and $R = u_x/2$ for the KK equations. The linear spectral problem gives rise to the Darboux transformations

$$\text{(SK)} \qquad \bar{u} - u = \partial_x^2 \log \Psi, \qquad \text{(KK)} \quad \bar{u} - u = \frac{1}{2}\partial_x^2 \log\left(\Psi\Psi_{xx} - \frac{1}{2}\Psi_x^2 + 3u\Psi^2\right). \tag{9}$$

It is interesting that the Bäcklund transformations (9) can be recast in the form

$$\bar{u} = u + y_1 - y_2^2, \qquad \bar{u} = u + \frac{1}{2}\partial_x \left[\frac{y_2 + 3\dot{u} + 6uy_1}{y_2 - y_1^2/2 + 3u}\right].$$

where y_1, y_2 are solutions to

$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}x} = y_2 - y_1^2, \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}x} = -6uy_1 + (6 - \lambda R) - y_1 y_2, \end{cases}$$

which is related to a Vessiot–Guldberg Lie algebra of vector fields $P_8 \simeq \mathfrak{sl}(3, \mathbb{R})$.

Bäcklund transformations for the KK and KS equations do not really depend on the linear spectral problem, but rather on the associated Riccati projective equations which contain all the necessary information for their description.

Riccati hierarchy and Gambier equations

The second-order differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{3}{4y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{3}{2}y^2\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{4}y^3 + 6uy - 2\lambda = 0, \qquad (10)$$

where $\lambda \in \mathbb{R}$ and u is an arbitrary x-dependent function, belongs to G25. For an arbitrary x-dependent function $u(x)$, this is not a Lie system when written as a first-order system by adding a new variable $v := \mathrm{d}y/\mathrm{d}x$. Indeed, the related x-dependent vector field

$$X = v\partial_y + [3v^2/(4y) - 3y^2v/2 + y^3/4 - 2\lambda)\partial_v + 6uy\partial_v.$$

gives rise, when $u(x)$ is not a constant function, to an infinite-dimensional Lie algebra V. In particular,

$$X_1 = v\partial_y + [3v^2/(4y) - 3y^2v/2 + y^3/4 - 2\lambda]\partial_v, \qquad X_2 = y\partial_v \in V,$$

and their successive Lie brackets make $\dim V = +\infty$. Meanwhile, the contact transformation

$$y := \frac{\lambda}{\mathrm{d}z/\mathrm{d}x + z^2/2 + 3u}$$

maps (10) into a second-order Riccati chain equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + 3z\frac{\mathrm{d}z}{\mathrm{d}x} + z^3 + 6uz + 3\frac{\mathrm{d}u}{\mathrm{d}x} - \lambda = 0.$$