# Measurable regularity properties of infinite-dimensional Lie groups

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**Reference:**

*Measurable regularity properties of infinite-dimensional Lie groups*, arXiv:1601.02568

**Parameter dependence of solutions to ODEs with measurabe right-hand sides:**

$M$ compact smooth manifold

$\mathcal{X}(M)$ Fréchet space of $C^\infty$-vector fields on $M$

For $y_0 \in M$, $\gamma \in L^1([0,1], \mathcal{X}(M))$, ODE

$$\begin{cases} y'(t) &= \gamma(t)(y(t)) \\ y(0) &= y_0; \end{cases}$$

solution $\eta_{y_0}$. If $\Phi^\gamma(t)(y_0) := \eta_{y_0}(t)$, then

$$\Phi^\gamma \colon [0,1] \to \mathrm{Diff}(M).$$

**Theorem** (G.'15) $\Phi^\gamma \in AC([0,1], \mathrm{Diff}(M))$, *and*

$L^1([0,1], \mathcal{X}(M)) \to AC([0,1], \mathrm{Diff}(M))$, $\gamma \mapsto \Phi^\gamma$

*is* $C^\infty$.

Likewise for $M$ non-compact, $\gamma \in L^1([0,1], \mathcal{X}_c(M))$

**Vector-valued Lebesgue spaces**

If $E$ is a Fréchet space, let $\mathcal{L}^1([a,b],E)$ be the space of all measurable maps $\gamma\colon [a,b]\to E$ such that

$$q\circ\gamma\in\mathcal{L}^1[a,b]$$

for each cts seminorm $q$ on $E$ and $\gamma([a,b])$ is separable.

**Vector-valued absolutely continuous functions**

Call $\eta\colon [a,b]\to E$ **absolutely continuous** if there exists $\gamma\in\mathcal{L}^1([a,b],E)$ such that

$$(\forall t\in[a,b])\qquad \eta(t)=\eta(a)+\int_a^t\gamma(s)\,ds.$$

Then $\eta'(t)=\gamma(t)$ almost everywhere

Get locally convex space $AC([a,b],E)$

**$\infty$-dim Lie group:** group $G$ with smooth manifold structure modelled on a locally convex space $E$ such that group operations are smooth; $\mathfrak{g} := L(G) := T_e G$

**Smooth maps** (Keller's $C^\infty_c$-maps):

$E$, $F$ lcx spaces, $U \subseteq E$ open

cts map $f\colon U \to F$ called **smooth** if iterated directional derivatives

$$(D_{y_n} \cdots D_{y_1} f)(x)$$

exist $\forall n$ & continuous in $(x, y_1, \ldots, y_n) \in U \times E^n$.

**Diffeomorphism groups**

$M$ compact smooth manifold

Diff($M$) group of $C^\infty$-diffeomorphisms of $M$

is an $\infty$-dim Lie group; parametrization of some open identity neighbourhood $U \subseteq$ Diff($M$):

$$\mathcal{X}(M) \supseteq V \to U \subseteq \text{Diff}(M), \quad X \mapsto \exp \circ X$$

where exp: $TM \to M$ is the exponential function for a Riemannian metric $g$ on $M$.

**Mapping groups**

Let $G$ be an $\infty$-dim Lie group modelled on a locally convex space $E$ and $\phi\colon G \supseteq U \to V \subseteq E$ a chart around $e$. Then

$$AC([0,1],V) \text{ is open in } AC([0,1],E)$$

and $AC([0,1],G)$ can be given a Lie group structure with

$$AC([0,1],U) \to AC([0,1],V), \quad \gamma \mapsto \phi \circ \gamma$$

as a chart around the neutral element $e$.

**General context: regularity of inf-dim Lie groups**

***G* acts on *TG*:** For $g \in G$, right translation $\rho_g \colon G \to G$, $x \mapsto xg$ is smooth; for $v \in TG$ set $v.g := T\rho_g(v)$

If $\gamma \colon [0,1] \to \mathfrak{g} = T_eG$ is smooth, then

$$\begin{cases} \eta'(t) &= \gamma(t).\eta(t) \\ \eta(0) &= e \end{cases}$$

has at most one solution $\eta \colon [0,1] \to G$; write $\mathrm{Evol}^r(\gamma) := \eta$

**Defn** (Milnor) If each smooth curve in $\mathfrak{g}$ has an evolution and

$$\mathrm{Evol}^r \colon C^\infty([0,1],\mathfrak{g}) \to C^\infty([0,1],G)$$

is smooth, then $G$ is called **regular**

Surprising fact: All known Lie groups regular

**Thm.** (Milnor'84) *Let $G$ and $H$ be Lie groups. If $G$ is 1-connected and $H$ is regular, then for every cts Lie algebra hom $\psi \colon \mathfrak{g} \to \mathfrak{h}$ there exists a smooth group hom $\phi \colon G \to H$ with $T_e\phi = \psi$.*

**Defn** Say a Fréchet-Lie group $G$ is **$L^1$-regular** if

$$\begin{cases} \eta'(t) &= \gamma(t).\eta(t) \\ \eta(0) &= e \end{cases}$$

has a (necessarily unique) solution $\mathrm{Evol}^r(\gamma) := \eta \in AC([0,1],G)$ for each $\gamma \in L^1([0,1],\mathfrak{g})$ and

$$\mathrm{Evol}^r\colon L^1([0,1],\mathfrak{g}) \to AC([0,1],G)$$

is smooth.

**Connection to first theorem on parameter-dependence:**

Diff$(M)$ has Lie algebra $\mathcal{X}(M)$; theorem says Diff$(M)$ is $L^1$-regular; evolution is

$$\mathrm{Evol}^r(\gamma) = \Phi^\gamma,$$

the flow of time-dependent vector field

$$\gamma \in L^1([0,1],\mathcal{X}(M))$$

**Further examples.**

(a) Every Banach-Lie group is $L^1$-regular (G.'15), e.g. $C^k(M,H)$ for $M$ compact, $k \in \mathbb{N}_0$, $H$ a Banach-Lie group

(b) $C^\infty(M,H) = \varprojlim C^k(M,H)$ is $L^1$-regular

And many more!

One application:

**Theorem** (G.'15) *If $G$ is $L^1$-regular, then $G$ has the Trotter property, i.e.,*

$$\lim_{n\to\infty} \Big( \exp_G(tv/n) \exp_G(tw/n) \Big)^n = \exp_G \Big( t(v+w) \Big)$$

*for all $v, w \in L(G)$, uniformly for $t$ in compact sets.*

## $L^1$-regularity of Banach-Lie groups

**Lemma A.** *Let $U$ be an open subset in a Fréchet space, $E$ be a Banach space and*

$$f\colon U \times E \to E$$

*be a smooth map which is linear in the second argument. Then the following map is smooth:*

$$f_*\colon C([0,1],U) \times L^1([0,1],E) \to L^1([0,1],E),$$

$$(\eta,\gamma) \mapsto f \circ (\eta,\gamma).$$

**Lemma B.** (Fixed points with parameters). *Let $P$ be an open set in a locally convex space, $B$ be a closed ball in a Banach space and*

$$f\colon P \times B \to B$$

be a smooth map such that the maps $f_p := f(p,.)\colon B \to B$ form a uniform family of contractions for $p \in P$, i.e.,

$$\sup_{p\in P} \operatorname{Lip}(f_p) < 1.$$

*Let $x_p \in B$ be the unique fixed point of $f_p$. Then $P \to B$, $p \mapsto x_p$ is smooth.*

If $\mathfrak{g}$ is a Banach space and the open 0-neighbhd $G \subseteq \mathfrak{g}$ a local Banach-Lie group, define

$$f\colon G \times \mathfrak{g} \to \mathfrak{g}$$

via $f(g,x) := d\rho_g(0,x)$. Let $\gamma \in L^1([0,1],\mathfrak{g})$. For $\eta$ in $C([0,1],G)$,

$$\eta'(t) = f(\eta(t),\gamma(t)), \quad \eta(0) = 0$$

can be rewritten as the integral equation

$$\eta = J(f_*(\eta,\gamma))$$

with $J\colon L^1([0,1],\mathfrak{g}) \to C([0,1],\mathfrak{g})$,

$$J(\theta)(t) := \int_0^t \theta(s)\,ds.$$

It can be solved for small $\gamma$ using Banach's Fixed Point Theorem (Picard Iteration). By Lemmas A and B, $\mathrm{Evol}^r(\gamma) = \eta$ is smooth in $\gamma$.

## $L^1$-regularity of diffeomorphism groups

$M$ compact $C^\infty$-manifold, $\gamma \in L^1([0,1], \mathcal{X}(M))$; use

$f\colon M \times \mathcal{X}(M) \to TM$, $(p, X) \mapsto X(p)$ is $C^\infty$.

For fixed $y_0 \in M$ can solve

$$\eta'(t) = \gamma(t)(\eta(t)) = f(\eta(t), \gamma(t)), \quad \eta(0) = y_0$$

as above using a Picard iteration for small $\gamma$; get $\eta = \eta_{y_0}$. Set $\Phi(t)(y_0) := \eta_{y_0}(t)$. Show

$$\Phi(t) \in \mathrm{Diff}(M), \quad \Phi \in AC([0,1], \mathrm{Diff}(M))$$

and $\Phi'(t) \circ \Phi(t)^{-1} = \gamma(t)$, i.e., $\mathrm{Evol}^r(\gamma) = \Phi$.

Using smooth parameter-dependence of fixed points (and exponential laws), show $\mathrm{Evol}^r$ is smooth.