The compact picture of symmetry breaking operators for rank one orthogonal and unitary groups

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50th Seminar Sophus Lie, Bedlewo

September 26, 2016

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$$(\mathrm{GL}(n,\mathbb{R}), \mathrm{GL}(n-1,\mathbb{R})), \quad (\mathrm{O}(p,q), \mathrm{O}(p,q-1)), \quad (\mathrm{U}(p,q), \mathrm{U}(p,q-1)), \quad \ldots$$

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Question

- In case 2, for which π, τ is the multiplicity $= 1$?
- If the multiplicity is $= 1$, construct explicitly a symmetry breaking operator $\pi|_{G'} \to \tau$.

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Idea: Study this question algebraically $\rightsquigarrow$ in a different category.

Harish-Chandra modules

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To each smooth admissible representation (π, V) of G one can associate the Harish-Chandra module $(\pi_{\mathrm{HC}}, V_{\mathrm{HC}})$ of $(\mathfrak{g}, K)$ on

$$V_{\mathrm{HC}} = \{v \in V : \dim \operatorname{span} \pi(K)v < \infty\} \simeq \bigoplus_{\sigma \in \widehat{K}} \underbrace{[\pi|_K : \sigma]}_{<\infty} \cdot \sigma \qquad (K\text{-finite vectors in } V).$$

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Theorem (Casselman–Wallach)

The functor $\pi \mapsto \pi_{\mathrm{HC}}$ is an equivalence of categories:

$$\left\{\begin{matrix}\text{smooth admissible Fréchet representations} \\ \text{of moderate growth}\end{matrix}\right\} \to \{\text{Harish-Chandra modules}\}$$

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Problem

For a smooth admissible representation π of G the restriction $\pi|_{G'}$ is in general not admissible.

$$\rightsquigarrow \text{ injective map:} \qquad \operatorname{Hom}_{G'}(\pi|_{G'}, \tau) \hookrightarrow \operatorname{Hom}_{(\mathfrak{g}', K')}(\pi_{\mathrm{HC}}|_{(\mathfrak{g}', K')}, \tau_{\mathrm{HC}}).$$

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Remark: Surjectivity of this map is equivalent to the automatic continuity of invariant distribution vectors for the homogeneous space $(G \times G')/\mathrm{diag}(G')$.

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Construct and classify algebraic symmetry breaking operators in

$$\mathrm{Hom}_{(\mathfrak{g}',K')}(\pi_{\mathrm{HC}}|_{(\mathfrak{g}',K')}, \tau_{\mathrm{HC}})$$

for irreducible Harish-Chandra modules π and τ of G and G' and multiplicity-one pairs (G, G').

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- For $\xi\in\widehat{M}$, $\xi'\in\widehat{M'}$ and $r,r'\in\mathbb{C}$ we consider the degenerate principal series representations

$$\pi_{\xi,r}=\mathrm{Ind}_P^G(\xi\otimes e^{r\nu}\otimes\mathbf{1}),\qquad \tau_{\xi',r'}=\mathrm{Ind}_{P'}^{G'}(\xi'\otimes e^{r'\nu'}\otimes\mathbf{1}).$$

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$$\rightsquigarrow \quad \text{Determine} \qquad \mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_{\xi,r})_{\mathrm{HC}}, (\tau_{\xi',r'})_{\mathrm{HC}})$$

Multiplicity-freeness assumptions

Decompose $\pi_{\xi,r}$ and $\tau_{\xi',r'}$ into K-isotypic components:

$$(\pi_{\xi,r})_{\mathrm{HC}} = \bigoplus_{\alpha\in\widehat{K}} \mathcal{E}(\alpha), \qquad (\tau_{\xi',r'})_{\mathrm{HC}} = \bigoplus_{\alpha'\in\widehat{K'}} \mathcal{E}'(\alpha').$$

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By 3 we can further decompose each K-type $\mathcal{E}(\alpha)$ into K'-representations:

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For every pair (α, α') with $\mathcal{E}(\alpha,\alpha') \neq 0$ and $\mathcal{E}'(\alpha') \neq 0$ we fix a K'-isomorphism

$$R_{\alpha,\alpha'} : \mathcal{E}(\alpha,\alpha') \xrightarrow{\sim} \mathcal{E}'(\alpha').$$

Scalar identities

By the multiplicity-free assumptions and Schur's Lemma, every linear map $T : (\pi_{\xi,r})_{\mathrm{HC}} \to (\tau_{\xi',r'})_{\mathrm{HC}}$ is K'-intertwining if and only if there exist scalars $t_{\alpha,\alpha'} \in \mathbb{C}$ such that

$$T|_{\mathcal{E}(\alpha,\alpha')} = \begin{cases} t_{\alpha,\alpha'} \cdot R_{\alpha,\alpha'} & \text{for } \mathcal{E}'(\alpha') \neq 0, \\ 0 & \text{else.} \end{cases}$$

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Theorem (M.–Ørsted 2014)

A linear map $T : (\pi_{\xi,r})_{\mathrm{HC}} \to (\tau_{\xi',r'})_{\mathrm{HC}}$ given by scalars $t_{\alpha,\alpha'} \in \mathbb{C}$ is $(\mathfrak{g}', K')$-intertwining if and only if for all $\alpha \in \widehat{K}$ and $\alpha', \beta' \in \widehat{K'}$ the following identity holds:

$$(\sigma'_{\beta'} - \sigma'_{\alpha'} + 2r')t_{\alpha,\alpha'} = \sum_{\substack{\beta \in \widehat{K} \\ (\alpha;\alpha') \leftrightarrow (\beta;\beta')}} \lambda^{\beta,\beta'}_{\alpha,\alpha'} (\sigma_\beta - \sigma_\alpha + 2r) t_{\beta,\beta'}. \qquad (\star)$$

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Proof: The LHS of $(\star)$ is obtained by first applying T to $\mathcal{E}(\alpha,\alpha')$ and then acting on $\mathcal{E}'(\alpha')$ by $\mathfrak{g}'$, the RHS is obtained by first acting on $\mathcal{E}(\alpha,\alpha')$ by $\mathfrak{g}'$ and then applying T.

- We write $(\alpha,\alpha')\leftrightarrow(\beta,\beta')$ if $\mathcal{E}(\beta,\beta')$ can be reached from $\mathcal{E}(\alpha,\alpha')$ by the action of $\mathfrak{g}'$.
- The numbers $\sigma_\alpha,\sigma'_{\alpha'}\in\mathbb{R}$ only depend on ξ,ξ' and α,α', not on $r,r'\in\mathbb{C}$, and can be computed easily from the highest weights of α,α'.
- The constants $\lambda^{\beta,\beta'}_{\alpha,\alpha'}$ depend only on the choice of the isomorphisms $R_{\alpha,\alpha'}$ and $R_{\beta,\beta'}$.

Example: $(G, G') = (\mathrm{O}(n+1,1), \mathrm{O}(n,1))$

- Let $(G, G') = (\mathrm{O}(n+1,1), \mathrm{O}(n,1))$ and consider spherical principal series representations

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- The maximal compact subgroups are $K = \mathrm{O}(n+1) \times \mathrm{O}(1)$ and $K' = \mathrm{O}(n) \times \mathrm{O}(1)$, and

$$(\pi_r)_{\mathrm{HC}}|_K \simeq \bigoplus_{\alpha=0}^{\infty} \underbrace{\mathcal{H}^{\alpha}(\mathbb{R}^{n+1}) \otimes \mathrm{sgn}^{\alpha}}_{\mathcal{E}(\alpha)}, \qquad (\tau_{r'})_{\mathrm{HC}}|_{K'} \simeq \bigoplus_{\alpha'=0}^{\infty} \underbrace{\mathcal{H}^{\alpha'}(\mathbb{R}^{n}) \otimes \mathrm{sgn}^{\alpha'}}_{\mathcal{E}'(\alpha')},$$

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- Classical branching rules:

$$\mathcal{H}^{\alpha}(\mathbb{R}^{n+1}) \otimes \mathrm{sgn}^{\alpha}\big|_{K'} \simeq \bigoplus_{\alpha'=0}^{\alpha} \underbrace{\mathcal{H}^{\alpha'}(\mathbb{R}^{n}) \otimes \mathrm{sgn}^{\alpha}}_{\mathcal{E}(\alpha,\alpha')}.$$

$$\Rightarrow \mathcal{E}(\alpha,\alpha') \simeq \begin{cases} \mathcal{E}'(\alpha') & \text{for } \alpha - \alpha' \text{ even}, \\ 0 & \text{for } \alpha - \alpha' \text{ odd}. \end{cases}$$

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- The maximal compact subgroups are $K = \mathrm{O}(n+1) \times \mathrm{O}(1)$ and $K' = \mathrm{O}(n) \times \mathrm{O}(1)$, and

$$(\pi_r)_{\mathrm{HC}}|_K \simeq \bigoplus_{\alpha=0}^{\infty} \underbrace{\mathcal{H}^\alpha(\mathbb{R}^{n+1}) \otimes \mathrm{sgn}^\alpha}_{\mathcal{E}(\alpha)}, \qquad (\tau_{r'})_{\mathrm{HC}}|_{K'} \simeq \bigoplus_{\alpha'=0}^{\infty} \underbrace{\mathcal{H}^{\alpha'}(\mathbb{R}^{n}) \otimes \mathrm{sgn}^{\alpha'}}_{\mathcal{E}'(\alpha')},$$

where $\mathcal{H}^k(\mathbb{R}^m)$ denotes the space of spherical harmonics on $\mathbb{R}^m$ of degree k.

- Classical branching rules:

$$\mathcal{H}^\alpha(\mathbb{R}^{n+1}) \otimes \mathrm{sgn}^\alpha\big|_{K'} \simeq \bigoplus_{\alpha'=0}^{\alpha} \underbrace{\mathcal{H}^{\alpha'}(\mathbb{R}^{n}) \otimes \mathrm{sgn}^\alpha}_{\mathcal{E}(\alpha,\alpha')}.$$

$$\Rightarrow \mathcal{E}(\alpha,\alpha') \simeq \begin{cases} \mathcal{E}'(\alpha') & \text{for } \alpha-\alpha' \text{ even,} \\ 0 & \text{for } \alpha-\alpha' \text{ odd.} \end{cases}$$

For $\alpha-\alpha'$ even we may choose $R_{\alpha,\alpha'} = \mathrm{rest}_{x_{n+1}=0} : \mathcal{H}^\alpha(\mathbb{R}^{n+1}) \xrightarrow{\sim} \mathcal{H}^{\alpha'}(\mathbb{R}^n)$.

Example: $(G, G') = (\mathrm{O}(n+1,1), \mathrm{O}(n,1))$

Scalar identities for $t_{\alpha,\alpha'}$

$$
\begin{aligned}
(2\alpha+n-2)(2r'+2\alpha'+n-2)t_{\alpha,\alpha'} &= (\alpha+\alpha'+n-2)(2r+2\alpha+n-1)t_{\alpha+1,\alpha'+1} \\
&\quad + (\alpha-\alpha')(2r-2\alpha-n+3)t_{\alpha-1,\alpha'+1} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
(2\alpha+n-2)(2r'-2\alpha'-n+4)t_{\alpha,\alpha'} &= (\alpha-\alpha'+1)(2r+2\alpha+n-1)t_{\alpha+1,\alpha'-1} \\
&\quad + (\alpha+\alpha'+n-3)(2r-2\alpha-n+3)t_{\alpha-1,\alpha'-1}. \qquad (2)
\end{aligned}
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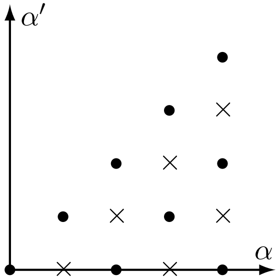
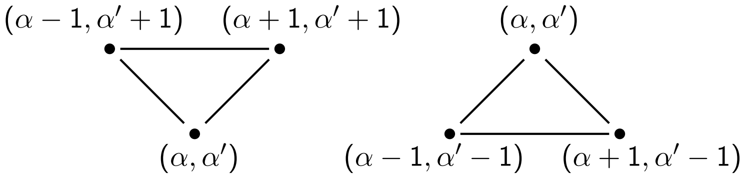
K-type picture:



Scalar identities:



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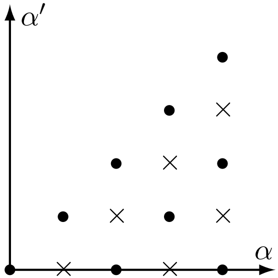

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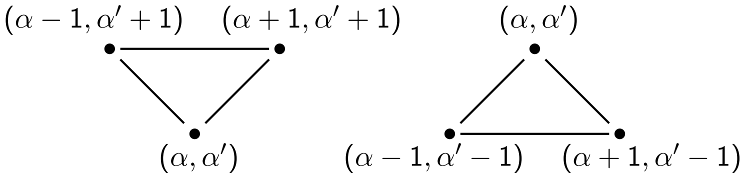

Theorem (M.–Ørsted 2014)

$$\dim \mathrm{Hom}_{(\mathfrak{g}',K')}((\pi_r)_{\mathrm{HC}}, (\tau_{r'})_{\mathrm{HC}}) = \begin{cases} 2 & \text{for } (r,r') = (-\frac{n}{2}-i, -\frac{n-1}{2}-j),\ i,j\in\mathbb{N},\ i-j\in 2\mathbb{N}, \\ 1 & \text{else.} \end{cases}$$

Example: $(G, G') = (O(n+1,1), O(n,1))$

Final remarks

- Multiplicity 2 in the previous theorem does not contradict the fact that $(O(n+1,1), O(n,1))$ is a multiplicity-one pair: For those (r, r') with multiplicity 2, the representations π_r and $\tau_{r'}$ are both reducible with two composition factors, and there exist independent intertwining operators between them.

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is a bijection.
- The method also works nicely for
  - $(G, H) = (U(n+1,1), U(n,1))$ and spherical principal series,
  - $(G, H) = (\mathrm{Pin}(n+1,1), \mathrm{Pin}(n,1))$ and spinor-valued principal series,
  - ...

Thank you for your attention!