Sophus Lie seminar

Bedlewo

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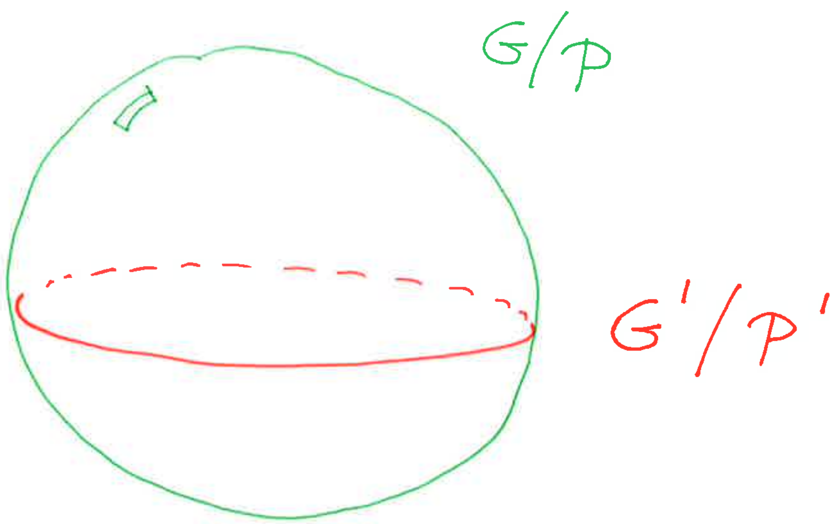


Elliptic boundary value problems and branching laws

Bent Ørsted
(joint with J. Möllers & G. Zhang)

IDEA Combine two topics:

(a) Poisson transforms for EBVP, symmetry breaking operators [T. Kobayashi]

(b) Branching laws for $G \supset G'$, π unitary representation of G, restrict $\pi|_{G'} \cong \int_{\widehat{G'}}^{\oplus} \pi'$

Classical example

$$F(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-s)^2 + y^2} f(s)\, ds$$

2)

$$(\mathrm{BVP}) \quad \left.\begin{aligned} \Delta F &= 0 \\ \lim_{y \downarrow 0} F(x,y) &= f(x) \end{aligned}\right\}$$

$$f(x) \xrightarrow{\text{Poisson } P} F(x,y)$$

also have a group action

$$g.x = \frac{ax+b}{cx+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

and the symmetry

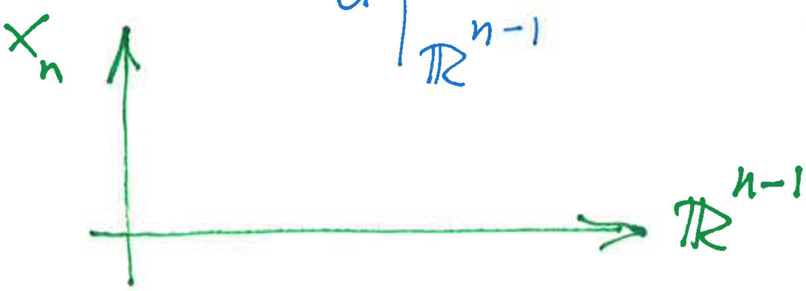
$$P \circ g^* = g^* \circ P, \quad g \in SL(2,\mathbb{R}) \subset SL(2,\mathbb{C})$$

Now

$$G = O(1, n+1) \quad \text{conformal } \mathbb{R}^n$$

$$G' = O(1,n) \quad \text{conformal } \mathbb{R}^{n-1}$$

$$(\mathrm{BVP}) \quad \left.\begin{aligned} \left(x_n^2 \Delta + a\, x_n \frac{\partial}{\partial x_n}\right) u &= 0 \\ u\big|_{\mathbb{R}^{n-1}} &= f \end{aligned}\right\} \text{[C.-S.]}$$



$(a \in \mathbb{R})$

3)

Tools (a) symmetry breaking = Poisson operators, $X' \subset X$

$$F(x) = \int_{X'} K(x, x') f(x')\,dx'$$

(b) branching theory for unitary spherical representations

$$\pi_\mu(g)F(x) = j(g^{-1}, x)^{\mu+\rho} F(g^{-1} x)$$

$$j(g, x) = |cx+d|^{-1}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\rho = \frac{n}{2} \quad \text{for } \mathbb{R}^n$$

Definition $\Delta_a = x_n^2 \Delta + a\, x_n \frac{\partial}{\partial x_n}$

$$\dot{H}^s(\mathbb{R}^n) = \left\{ \|u\|^2 = \int |\hat{u}(\xi)|^2 |\xi|^{2s} d\xi < \infty \right\}$$

Thm A Δ_a is selfadjoint in $\dot{H}^s(\mathbb{R}^n)$ with explicit spectrum and we have $\dot{H}^s(\mathbb{R}^n) \to \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ a unitary G'-isomorphism from solutions to boundary values.

4)

Thm B "CR case"

Thm C "General kernels" for principal series $G \supset G'$ symmetric

[J. Möllers, Y. Oshima, —]

Details (1) spectrum of Δ_a in $\dot{H}^s$, $s = \frac{2-a}{2}$, $2-n < a \leq 2$ is

$$\sigma_p = \{ k(k+a-1) \mid k \in \mathbb{N},\ k < \tfrac{1-a}{2} \}$$

$$\sigma_c = (-\infty, -(\tfrac{1-a}{2})^2)$$

(2) $2-n < a < 1$; for $f \in \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ there is a unique $u \in \dot{H}^s(\mathbb{R}^n)$ with $\Delta_a u = 0$, $u|_{\mathbb{R}^{n-1}} = f$ and

$$u(x) = P_a f(x) = c_{n,a} \int_{\mathbb{R}^{n-1}} \frac{|x_n|^{1-a} f(y)\,dy}{(|x'-y|^2 + x_n^2)^{\frac{n-a}{2}}}$$

$$c_{n,a} = \frac{\Gamma(\frac{n-a}{2})}{\pi^{\frac{n-1}{2}} \Gamma(\frac{1-a}{2})}, \quad x = (x', x_n)$$

$$\|u\|^2 = [2^a \pi \Gamma(2-a) / \Gamma(\tfrac{1-a}{2}) \Gamma(\tfrac{3-a}{2})] \|f\|^2$$

5)

(3) $2-n < a < -1$; for $g \in \dot{H}^{s-1}(\mathbb{R}^{n-1})$ there is a unique $u \in \dot{H}(\mathbb{R}^n)$ with

$$\Delta_a u = a u \,, \quad \frac{\partial}{\partial x_n} u \Big|_{\mathbb{R}^{n-1}} = g \quad \text{and}$$

$$u(x) = Q_a g(x) = c'_{n,a} \int_{\mathbb{R}^{n-1}} \frac{x_n |x_n|^{-1-a} g(y)\,dy}{(|x'-y|^2 + x_n^2)^{\frac{(n-a-2)}{2}}}$$

(4) $$\left.\begin{aligned} \Delta_a u &= k(k+a-1)u \\ D_{a,k} u\Big|_{\mathbb{R}^{n-1}} &= f \end{aligned}\right\} 0 \le k < \frac{1-a}{2}$$

$$D_{a,k} u = p\left(\Delta', \frac{\partial}{\partial x_n}\right) u \,, \quad \text{pol. degree } 2k$$

$$u(x) = P_{a,k} f(x) = c_{n,a,k} \int_{\mathbb{R}^{n-1}} \frac{x_n^k |x_n|^{1-a-2k} f(y)\,dy}{(|x'-y|^2 + x_n^2)^{\frac{n-a}{2}-k}}$$

$$\left(f \in \dot{H}^{\frac{1-a}{2}-k}(\mathbb{R}^{n-1}) \right)$$

Group actions

$$G \supset P = MAN \,, \quad N \simeq \mathbb{R}^n$$

$$G' \supset P' = M'A'N' \,, \quad N' \simeq \mathbb{R}^{n-1}$$

$$\pi_\mu = \mathrm{Ind}_P^G(\chi_\mu) \,, \quad \pi_\nu = \mathrm{Ind}_{P'}^{G'}(\chi_\nu)$$

6)

action on $\dot{H}^s(\mathbb{R}^n)$ has $\mu = \frac{a-2}{2} = -s$

action on $\dot{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$ has $\nu = \frac{a-1}{2} = -s+\frac{1}{2}$

Main fact

$$\boxed{\begin{aligned} d\pi_\mu(\mathrm{Cas}_{G'}) &= x_n^2 \Delta + 2(\mu+1) x_n \frac{\partial}{\partial x_n} \\ &\quad + (\mu+\rho)(\mu-\rho+1) \end{aligned}}$$

Hence our Poisson transforms = symmetry breaking operators give unitary intertwining operators from the discrete spectrum in the branching law and boundary values.

Hardy-Littlewood-Sobolev $\frac{1}{2} < s = \frac{2-a}{2} < \frac{n}{2}$

$$\begin{array}{ccc} \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^{n-1}) & \xrightarrow{[\text{M.O.Ø.}]} & \dot{H}^s(\mathbb{R}^n) \\ \downarrow & \mathcal{P}_a & \downarrow \\ L^p(\mathbb{R}^{n-1}) & \xrightarrow[\text{[S. Chen]}]{} & L^q(\mathbb{R}^n) \end{array}$$

BEST CONSTANTS

$p = \frac{2(n-1)}{n-2-a}$ $\qquad$ $q = \frac{2n}{n-2+a}$

7)

Remark Corresponding bilinear forms

$$\ell^{mod}_{\mu,\nu}(F,f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n-1}} (|x'-y|^2 + x_n^2)^{\nu-\rho'} \cdot |x_n|^{\mu-\rho-\nu+\rho'} F(x) f(y)\,dy\,dx$$

has simple poles at $\mu+\rho\pm\nu-\rho' \in -2\mathbb{N}$

unique outside as invariant for

$\pi_\mu|_{G'} \otimes \pi_\nu$ diagonal action [M.O.Ø.]

Application [M.Ø.] to compact hyperbolic manifolds

$$\left\{ \begin{array}{l} Y'^m \subset Y^n \text{ geodesic} \\ \phi \text{ automorphic function on } Y \end{array} \right.$$

$\Downarrow$

$c_j = \int_{Y'} \phi \overline{\phi_j}$ satisfies

$$\sum_{|\lambda_j| \le T} b_j \le C \cdot T^{\frac{2n-m-3}{2}}$$

$\{\lambda_j\} = \operatorname{spec} \Delta_{Y'}$,

where $b_j = |c_j|^2 e^{\pi\sqrt{\lambda_j}}$ ref Berns.-Rez.

8)

CR - case $\quad G = SU(1, n+1)$, $N \cong$

Heisenberg group $\mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R}$

$(z,t)(\tilde{z},\tilde{t}) = (z+\tilde{z}, t+\tilde{t}+2\,\mathrm{Im}\, z\overline{\tilde{z}})$

Let $$\mathcal{L} = \sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}\right)^2 + \left(\frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}\right)^2$$

$$\mathcal{L}_a = |z_n|^2 \mathcal{L} + a\left(x_n \frac{\partial}{\partial x_n} + y_n \frac{\partial}{\partial y_n}\right)$$

Thm B (1) For $-2n < a \leq 2$, $\mathcal{L}_a$ is essentially selfadjoint on $\dot{H}^s(\mathbb{H}^n)$, $s = \frac{2-a}{2}$, with eigenvalues

$$2k(2k+a), \quad k \in \mathbb{N}, \quad 0 \leq 2k < -\frac{a}{2}$$

(2) For $-2n < a < 0$, $f \in \dot{H}^{s-1}(\mathbb{H}^{n-1})$ has a unique $u \in \dot{H}^s(\mathbb{H}^n)$ with

$$\mathcal{L}_a u = 0, \quad u|_{\mathbb{H}^{n-1}} = f \quad \text{and} \quad [\text{NEW}]$$

it is $$P_a f(z,t) = c \cdot \int_{\mathbb{H}^{n-1}} \frac{|z_n|^{-a} f(z',t')}{|(z,t)^{-1}(z',t')|^{2n-a}}\, dz'dt'$$

— isometry up to a constant

recall norm $|(z,t)| = (|z|^4 + t^2)^{1/4}$.

9)

(3) similarly for other values of k

$$D_{a,k} : \dot{H}^{s}(\mathbb{H}^n) \longrightarrow \dot{H}^{s-1-2k}(\mathbb{H}^{n-1})$$

explicit differential operators

[like Juhl's operators in the real case]

NB again equivariant for the action of $G' = su(1,n)$ viz. principal series

Recall rank one groups $G = U(1, n+1; \mathbb{F})$

$$\mathrm{Ind}_P^G(1 \otimes e^{\mu} \otimes 1) =$$

$$\{ f \in C^\infty(G) \mid f(gman) = a^{-\mu-\rho} f(g) \}$$

giving (π_μ, I_μ) with $S(\bar{N}) \subset I_\mu \subset C^\infty(\bar{N})$

$\rho = \frac{n}{2}$ resp. $n+1$, $2n+3$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

unitarizable for $\mu \in i\mathbb{R}$ or

$$\mu \in \begin{cases} (-\rho, \rho) & \mathbb{F} = \mathbb{R}, \mathbb{C} \\ (-(\rho-2), \rho-2) & \mathbb{F} = \mathbb{H}. \end{cases}$$

$G' = U(1, n; \mathbb{F}) \times U(1; \mathbb{F})$

giving (τ_ν, J_ν) similarly.

Now symmetry breaking operators

$A_{\mu,\nu} : I_\mu \to J_\nu$

$A_{\mu,\nu} u(n') = \int_{\bar N} K_{\mu,\nu}(n,n') u(n)\, dn$

$K_{\mu,\nu}((z,t),(z',t')) = |(z,t)^{-1}(z',t')|^{-2(\nu+\rho')} \cdot |z_n|^{\mu-\rho+\nu+\rho'}$

and the transpose

$B_{\mu,\nu} = A^T_{-\mu,-\nu} : \mathcal{H}'^{-\infty}_\nu \to \mathcal{H}^{-\infty}_\mu$

$B_{\mu,\nu} f(n) = \int_{\bar N'} K_{-\mu,-\nu}(n,n') f(n')\, dn'$

Fact these are G'-maps

Extension to other pairs [M.O.Ø.]

semisimple $G \supset H$ symmetric $= G^\sigma$

parabolic $P = MAN \subset G$

$P \cap H = P_H$, $w_0^{-1} P w_0 = \bar P$

$w_0^{-1} P_H w_0 = \bar P_H$, $\bar N MAN \xrightarrow{a} A$

Thm C For $\alpha, \beta \in \mathfrak{a}_{\mathbb{C}}^*$ the kernels $\boxed{K_{\alpha,\beta}(g,h) = a(w_0^{-1}g^{-1}h)^{\alpha}\, a(w_0^{-1}g^{-1}\sigma(g))^{\beta}}$ define a meromorphic family of symmetry breaking ($= H$ equivariant) operators $A : I_\mu \longrightarrow J_\nu$ where

$$\mu = -w_0\alpha + \sigma\beta - w_0\beta + \rho$$
$$\nu = \alpha\big|_{\mathfrak{a}_{\mathbb{C}}^{\sigma}} + \rho'$$

and they are generically unique.

Note Special cases are Knapp-Stein operators ($G = H$) and trilinear functionals ($H = \mathrm{diag}(G \times G) \subset G \times G$) [Clerc] [Kobayashi - Speh]

Finally A formula that combines the boundary value problem, the spectrum of Δ_a, and the branching law (real case):

12) Let

$$D_{a,j} f(y) = \frac{j!}{2^j \left(\frac{a-1}{2}\right)_j} C_j^{\frac{a-1}{2}}\left(-\Delta', \frac{\partial}{\partial x_n}\right) f(y,0)$$

$$P_{a,j} F(x) = \frac{\Gamma\left(\frac{n-a}{2}-j\right)}{j!\, \pi^{\frac{n-1}{2}} \Gamma\left(\frac{1-a}{2}-j\right)} \int_{\mathbb{R}^{n-1}} \frac{x_n^j |x_n|^{1-a-2j}}{\left(|x'-y|^2+x_n^2\right)^{\frac{n-a}{2}-j}} F(y)\, dy$$

$$D_{a,\nu,\varepsilon} f(y) = \int_{\mathbb{R}^n} \frac{(\operatorname{sgn} x_n)^\varepsilon |x_n|^{\frac{a-3}{2}+i\nu}}{\left(|x'-y|^2+x_n^2\right)^{\frac{n-1}{2}+i\nu}} f(x)\, dx$$

$$P_{a,\nu,\varepsilon} F(x) = \int_{\mathbb{R}^{n-1}} \frac{(\operatorname{sgn} x_n)^\varepsilon |x_n|^{\frac{1-a}{2}-i\nu}}{\left(|x'-y|^2+x_n^2\right)^{\frac{n-1}{2}-i\nu}} F(y)\, dy$$

then for $2-n < a < 1$ we have

$$\boxed{\begin{aligned} f(x) &= \sum_{j\in[0,\frac{1-a}{2})\cap\mathbb{Z}} P_{a,j} D_{a,j} f(x) \\ &+ \frac{1}{4\pi^n} \sum_{\varepsilon=0,1} \int_0^\infty P_{a,\nu,\varepsilon} D_{a,\nu,\varepsilon} f(x) \cdot \left|\frac{\Gamma\left(\frac{n-1}{2}+i\nu\right)}{\Gamma(i\nu)}\right|^2 d\nu \end{aligned}}$$

13)

$$\|f\|^2_{\dot H^{\frac{2-a}{2}}(\mathbb{R}^n)} = \sum_{j\in[0,\frac{1-a}{2})\cap\mathbb{Z}} \alpha_{a,j}\,\|D_{a,j}f\|^2_{\dot H^{\frac{1-a}{2}-j}(\mathbb{R}^{n-1})}$$

$$+\frac{1}{2^a\pi^n}\sum_{\varepsilon=0,1}\int_0^\infty \|D_{a,\nu,\varepsilon}f\|^2_{L^2(\mathbb{R}^{n-1})}\,\beta_{a,\nu,\varepsilon}\,d\nu$$

$$\alpha_{a,j} = \frac{2^{a+2j}\pi\,\Gamma(2-a-j)}{j!\,(\frac{1-a}{2}-j)\,\Gamma(\frac{1-a}{2}-j)^2}$$

$$\beta_{a,\nu,\varepsilon} = \left|\frac{\Gamma((3-a+2\varepsilon+2i\nu)/4)\,\Gamma(\frac{n-1}{2}+i\nu)}{\Gamma((a-1+2\varepsilon+2i\nu)/4)\,\Gamma(i\nu)}\right|^2$$

Gegenbauer polynomials above:

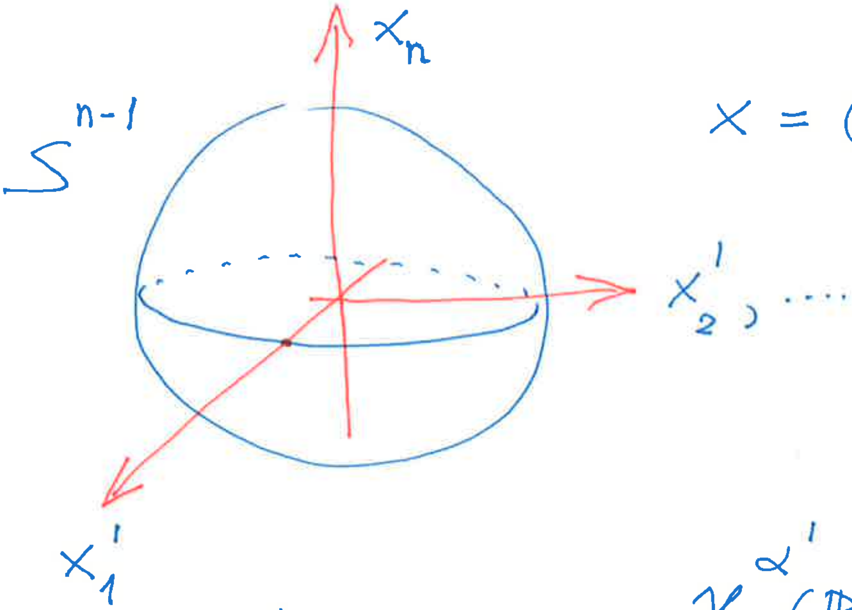
$$C_n^\alpha(z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(\lambda)_{n-k}\,(2z)^{n-2k}}{k!\,(n-2k)!} \qquad (\lambda=\alpha)$$

$$C_n^\alpha(x,y) = x^{n/2}\,C_n^\alpha(y/\sqrt{x}).$$

Further aspects:

(1) study such Poisson transforms more generally

(2) Study also the "compact picture" of the induced representations $G/P = K/K\cap P$ e.g. for the real case [M. Ø.] $(n+1 \longleftrightarrow n)$



$x = (x', x_n)$

ϕ spherical harmonic $\mathcal{H}^{\alpha'}(\mathbb{R}^{n-1})$

e.g. "Funk formula"

$$\boxed{\begin{aligned}&\int_{S^{n-1}} (|x'-y|^2 + x_n^2)^{-(r'+\frac{n-2}{2})} |x_n|^{r+r'-\frac{1}{2}} \\ &\qquad \cdot \phi(x') \, C^{\alpha'+\frac{n-2}{2}}_{\alpha-\alpha'}(x_n)\, dx \\ &= c_{\alpha,\alpha'}(r,r') \cdot \phi(y)\end{aligned}}$$

NB $r' = r + \frac{1}{2} + 2N$ gives $D_{2N}(r)$ Juhl's