Resonances and singular integrals

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Context: Geometric Scattering Theory

Spectral analysis of the (positive) Laplacian Δ on a complete non-compact Riemannian manifold (X, g).

Examples:

- Euclidean space $\mathbb{R}^n$: $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$.
- Poincaré half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ with hyperbolic metric: $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$

Δ is a positive, essentially self-adjoint operator on the Hilbert space $L^2(X)$.
Suppose: Δ has continuous spectrum $\sigma(\Delta) = [\rho_X^2, +\infty[$ with $\rho_X^2 \geq 0$.

The resolvent of Δ

$$R_\Delta(u) = (\Delta - u)^{-1}$$

is a bdd operator on $L^2(X)$ depending holomorphically on $u \in \mathbb{C} \setminus \sigma(\Delta)$, i.e.

$$\mathbb{C} \setminus \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) \in \mathcal{B}(L^2(X)).$$

is a holomorphic operator-valued function.

As operator on $L^2(X)$, the resolvent R_Δ has no extension across $\sigma(\Delta)$.

Letting R_Δ act on a smaller dense subspace of $L^2(X)$, e.g. $C_c^\infty(X)$, a meromorphic continuation of R_Δ across $\sigma(\Delta)$ is possible in many cases.

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Problem 1: Meromorphic continuation

Wanted: meromorphic continuation of $R_\Delta : \mathbb{C} \setminus \sigma(\Delta) \longrightarrow \mathcal{B}(L^2(X))$ across $\sigma(\Delta)$, by replacing $\mathcal{B}(L^2(X))$ with $\mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$

i.e.

- a Riemann surface $\begin{array}{c} M \\ \downarrow{\scriptstyle \pi} \\ \Omega \end{array}$ with Ω open in $\mathbb{C}$, containing (a part of) $\sigma(\Delta)$
- $\widetilde{R}_\Delta : M \to \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$ meromorphic and extending a lift of R_Δ to M:

$$\begin{tikzcd} M \arrow[r, "\widetilde{R}_\Delta"] & \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)') \\ \Omega \setminus \sigma(\Delta) \arrow[u, hook] \arrow[ur, "R_\Delta"'] & \end{tikzcd}$$

$\forall f, g \in C_c^\infty(X)$: $\langle \widetilde{R}_\Delta(\cdot) f, g \rangle_{L^2(X)}$ lifts and extends to M the function $\langle R_\Delta(\cdot) f, g \rangle_{L^2(X)}$

If this is possible:
The poles of $\widetilde{R}_\Delta$ are called the resonances of Δ.

Problem 2: Localization and nature of the resonances

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$\widetilde{R}_\Delta : M \to \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$ mero extension of R_Δ.
To simplify notation, suppose: $\widetilde{R}_\Delta : M \to \mathrm{Hom}(C_c^\infty(X), C^\infty(X))$
(true if X is a Riemannian symmetric space of the noncompact type)

Let z_0 be a resonance of Δ.
The residue operator at z_0 is the linear operator

$$\mathrm{Res}_{z_0} \widetilde{R}_\Delta : C_c^\infty(X) \to C^\infty(X)$$

"defined" for $f \in C_c^\infty(X)$ by

$$\mathrm{Res}_{z_0} \widetilde{R}_\Delta(f) : X \ni y \longrightarrow \mathrm{Res}_{z=z_0} \left[\widetilde{R}_\Delta(z)(f)\right](y) \in \mathbb{C}$$

["defined": residues are computed wrt charts in M, so up to nonzero constant multiples]

Well-defined: the subspace $\mathrm{Res}_{z_0} := \widetilde{R}_\Delta\left(C_c^\infty(X)\right)$ of $C^\infty(X)$.

The rank of the residue operator at z_0 is $\dim\left(\mathrm{Res}_{z_0}\right)$.

Problem 3: Find image and rank of the residue operator at z_0

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Additional properties appear when X is a homogenous space of a Lie group G endowed with a G-invariant Riemannian metric.

Example:
$X = G/K$ is a Riemannian symmetric space of the noncompact type, where:
G connected noncompact real semisimple Lie group with finite center
K maximal compact subgroup of G

e.g.

- Poincaré half-plane $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$; more generally $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$
- real hyperbolic space $H^n(\mathbb{R}) = \mathrm{SO}_0(1,n)/\mathrm{SO}(n)$

The Laplacian Δ of X is G-invariant

$\rightsquigarrow$ $R_\Delta(z)$ and its mero extension $\widetilde{R}_\Delta(z)$ are G-invariant

$\rightsquigarrow$ the residue operator at a resonance z_0 is a G-invariant op : $C_c^\infty(X) \to C^\infty(X)$

$\rightsquigarrow$ its image $\mathrm{Res}_{z_0} \subset C^\infty(X)$ is a G-module

(a K-spherical rep of G if $X = G/K$ is Riem. symmetric noncompact type)

Problem 3′: Which (spherical) representations of G we obtain?

Rank of residue operator $\equiv$ dimension of the corresponding representation
Irreducible? Unitary?

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Some usual renormalizations

In the literature on resonances, the setting is usually normalized as follows:

- Translate the spectrum $[\rho_X^2, +\infty)$ to $[0, +\infty)$
  i.e. consider $\Delta - \rho_X^2$ instead of Δ
- Change variables $u = z^2$ $\rightsquigarrow$ choice of square root: $\sqrt{-1} = i$
  $u \in \mathbb{C} \setminus [0, +\infty[$ corresponds to $z \in \mathbb{C}^+ = \{w \in \mathbb{C} : \mathrm{Im}\, w > 0\}$.
- Define

$$R(z) = R_{\Delta - \rho_X^2}(z^2) = (\Delta - \rho_X^2 - z^2)^{-1}$$

So $R : \mathbb{C}^+ \to \mathcal{B}(L^2(X))$ is a holomorphic operator-valued function.

Wanted:

Meromorphic continuation across $\mathbb{R}$ of $R : \mathbb{C}^+ \to \mathrm{Hom}\,(C_c^\infty(X), C_c^\infty(X)')$

$C^\infty(X)$ instead of $C_c^\infty(X)'$ for $X = G/K$ symmetric

In the following:

◇ the meromorphic extension of R is denoted by $\widetilde{R}$,

◇ a resonance of Δ is a pole of $\widetilde{R}$.

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The classical example $(\mathbb{R}^n, \Delta_{\mathbb{R}^n} = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2})$

The resolvent $R : \mathbb{C}^+ \to \mathrm{Hom}\left(C_c^\infty(\mathbb{R}^n), C^\infty(\mathbb{R})\right)$ extends:

- ⋄ holomorphically to $\mathbb{C}$ if $n \geq 3$ is odd,
- ⋄ holomorphically to a logarithmic cover of $\mathbb{C}$ if $n \geq 2$ is even,
- ⋄ meromorphically to $\mathbb{C}$ with unique simple pole (resonance) at $z = 0$ if $n = 1$.

If $\mathbf{n} = \mathbf{1}$:

- ⋄ the residue operator at $z = 0$ is

$$\mathrm{Res}_0 \widetilde{R} : C_c^\infty(\mathbb{R}) \ni f \longrightarrow \begin{bmatrix} \text{the constant function} \\ \mathbb{R} \ni y \to \widehat{f}(0) \in \mathbb{C} \end{bmatrix} \in C^\infty(\mathbb{R})$$

- ⋄ $\mathrm{Res}_0 := \mathrm{Res}_0 \widetilde{R}\big(C_c^\infty(\mathbb{R})\big) = \mathbb{C}$,
- ⋄ $\mathbb{R}^n \curvearrowright \mathrm{Res}_0$ is the trivial representation.

Interesting: Riemann surface of the extension of R of different type according to even/odd dimensions.

Resonances appear for Schrödinger operators (or Hamiltonians) $H = \Delta_{\mathbb{R}^n} + V$ where V is a potential acting as a multiplication operator.

$\rightsquigarrow$ suitable assumptions on V ensure that H extends as an ess s.a. op on $L^2(\mathbb{R}^n)$ with continuous spectrum $[0, +\infty[$:

e. g.: H ess s.a if V real valued; spectrum is $[0, +\infty[$ if $\lim_{|x|\to\infty} V(x) = 0$

The notion of resonance originated ($\sim$ 1930s) in Quantum Mechanics for Schrödinger operators: resonances are the metastable stable states of a system of Hamiltonian H.

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The real hyperbolic space $H^n(\mathbb{R})$

Motivation: Geometric Scattering

e.g. $X = \Gamma \setminus H^n$, asymptotically hyperbolic manifold, quotient of the real hyperbolic space by a suitable discrete subgroup Γ of $\mathrm{SO}_0(1, n)$.
Δ_X has a continuous spectrum $[\rho_n^2, +\infty)$ and a finite point spectrum.
Resonances of Δ_X are related to the dynamical Zeta function and completely characterize the length of the closed geodesics of X [Guillopé-Zworski, Patterson-Perry, Borthwick-Perry, Guillarmou-Naud...]

The case of $H^n(\mathbb{R})$

- L. Guillopé and M. Zworski (1995):
  - ◇ n odd: no resonances.
  - ◇ n even: infintely many resonances. Residue operators have **finite rank**.
- M. Zworski (2006): image of the residue ops related to spherical harmonics.

$H^n(\mathbb{R}) = \mathrm{SO}_0(1, n)/\mathrm{SO}(n)$ is the simplest family of noncompact symm. spaces

Why studying resonances on symmetric spaces?

- ◇ well understood geometry
- ◇ well developed Fourier analysis: HF (=Helgason-Fourier) transform
- ◇ radial part of Δ on a Cartan subspace is a Schrödinger operator
- ◇ tools from representation theory

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- ◇ tools from representation theory

The real hyperbolic space $H^n(\mathbb{R})$

Motivation: Geometric Scattering

e.g. $X = \Gamma \setminus H^n$, asymptotically hyperbolic manifold, quotient of the real hyperbolic space by a suitable discrete subgroup Γ of $\mathrm{SO}_0(1,n)$.
Δ_X has a continuous spectrum $[\rho_n^2, +\infty)$ and a finite point spectrum.
Resonances of Δ_X are related to the dynamical Zeta function and completely characterize the length of the closed geodesics of X [Guillopé-Zworski, Patterson-Perry, Borthwick-Perry, Guillarmou-Naud...]

The case of $H^n(\mathbb{R})$

- L. Guillopé and M. Zworski (1995):
- ◇ n odd: no resonances.
- ◇ n even: infintely many resonances. Residue operators have **finite rank**.
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Riemannian symmetric spaces of noncompact type $X = G/K$

General X of real rank one:

- R. Miatello and C. Will (2000):

meromorphic continuation of the resolvent (in the context of Damek-Ricci spaces).

- J. Hilgert and A.P. (2009):

meromorphic continuation of the resolvent (using HF transform).

◇ (infinitely many) resonances for $X \neq H^n(\mathbb{R})$ with n odd.

◇ **Finite rank** residue operators, image: irreducible finite dim K-spherical reps of G.

X of real rank ≥ 2:

- R. Mazzeo and A. Vasy (2005), A. Strohmaier (2005):

◇ analytic continuation of the resolvent of Δ from $\mathbb{C}^+$ across $\mathbb{R}$

$$\begin{cases} \text{to an open domain in } \mathbb{C}, & \text{if the real rank of } X \text{ is odd} \\ \text{to a logarithmic cover of an open domain in } \mathbb{C}, & \text{if the real rank of } X \text{ is even} \end{cases}$$

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◇ **If any**, resonances are along the negative imaginary axis.

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The resolvent of Δ on $X = G/K$

Explicit formula for the resolvent $R(z)$ of Δ on $C_c^\infty(X)$ via HF transform:

for $z \in \mathbb{C}^+$

$$R(z) = (\Delta - \rho_X^2 - z^2)^{-1} : f \in C_c^\infty(X) \to R(z)f \in C^\infty(X)$$

is given by

$$[R(z)f](y) \asymp \int_{\mathfrak{a}^*} \frac{1}{\langle \lambda, \lambda \rangle - z^2} (f \times \varphi_{i\lambda})(y) \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \qquad (y \in X),$$

where

$\mathfrak{a}^*$ = dual of a Cartan subspace $\mathfrak{a}$ $\rightsquigarrow$ real rank of X := dim $\mathfrak{a}^*$

$\langle \cdot, \cdot \rangle$ = inner product on $\mathfrak{a}^*$ induced by the Killing form of the Lie algebra of G

$\rightsquigarrow$ extend $\langle \cdot, \cdot \rangle$ to the complexification $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}^*$ by $\mathbb{C}$-bilinearity

φ_λ = spherical function on X of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$\rightsquigarrow$ the spherical functions on X are:

- the (normalized) K-invariant joint eigenfunctions of the commutative algebra of G-invariant diff ops on X
- matrix coefficients of the principal K-spherical reps of G corresponding to 1_K

$f \times \varphi_{i\lambda}$ = convolution on X of f and $\varphi_{i\lambda}$

$\rightsquigarrow$ by the Paley-Wiener thm for the HF-transform: entire and rapidly decreasing in $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

$c(\lambda)$ = Harish-Chandra's c-function

$\frac{1}{c(i\lambda)c(-i\lambda)}$ = Plancherel density for the HF-fransform

The resolvents of the Laplacians of $\mathbb{R}^n$ and X have similar structure:

Resolvent of the Laplacian on $\mathbb{R}^n$

$$[R(z)f](y) \asymp \int_{\mathbb{R}^n} \frac{1}{|\lambda|^2 - z^2}\, e^{iy\cdot\lambda}\widehat{f}(\lambda)\, d\lambda \qquad (f \in C_c^\infty(\mathbb{R}^n)\,, y \in \mathbb{R}^n)$$

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$\mathbb{R}^n$	$\longleftrightarrow$	$\mathfrak{a}^*$
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$e^{iy\cdot\lambda}\widehat{f}(\lambda)$	$\longleftrightarrow$	$(f \times \varphi_{i\lambda})(y)$ $\longrightarrow$ $\boxed{=\varphi_{i\lambda}(y)[HF \text{ transform of } f](i\lambda) \text{ if } f \text{ is right-}K\text{-invariant}}$
$d\lambda$	$\longleftrightarrow$	$\dfrac{d\lambda}{c(i\lambda)c(-i\lambda)}$

Difference:

In general, the Plancherel density for X is a meromorphic function of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$
$\rightsquigarrow$ these singularities *might* originate resonances

Remark: "might" :

- Plancherel density is nonsingular ($\Leftrightarrow$ even multiplicity case): then no resonances
- Plancherel density might be singular, and still no resonances

  e.g. $H^n(\mathbb{R}) \times X$ where n odd and X of rank one

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Some structure: roots of G/K

$\mathfrak{a} \curvearrowright \mathfrak{g}$ (=Lie algebra of G) by adjoint action $\mathrm{ad}\, H$ with $H \in \mathfrak{a}$.

e.g. If $\mathfrak{g} \subset \mathrm{Mat}(n, \mathbb{R})$, then $\mathrm{ad}\, H(X) = [H, X] = HX - XH$

$\{\mathrm{ad}\, H : H \in \mathfrak{a}\}$ commuting family of semisimple linear endomorphisms of $\mathfrak{g}$

Σ = non-zero joint eigenvalues of $\{\mathrm{ad}\, H : H \in \mathfrak{a}\}$= roots of $(\mathfrak{g}, \mathfrak{a})$

$\rightsquigarrow \Sigma$ is a finite subset of $\mathfrak{a}^*$

Σ^+ = choice of positive positive roots in Σ

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$ = root space of $\alpha \in \Sigma$

$m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ = multiplicity of the root α

$\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$

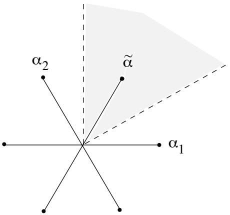
Example: $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$

$\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ = 3×3 matrices with real coeffs and trace 0

$\mathfrak{a} = \{H = \mathrm{diag}(h_1, h_2, -(h_1 + h_2)) : h_1, h_2 \in \mathbb{R}\} \cong \mathbb{R}^2$

In this case:

Σ of type A_2
$\Sigma^+ = \{\alpha_1, \alpha_2, \widetilde{\alpha} = \alpha_1 + \alpha_2\}$
$m_\alpha = 1$ for all α



The Plancherel density $[c(i\lambda)c(-i\lambda)]^{-1}$

Notation: For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\alpha \in \Sigma$ set $\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$

Harish-Chandra c-function:

$\Sigma_*^+ = \{\beta \in \Sigma^+ : 2\beta \notin \Sigma\}$ (the unmultipliable positive roots)

$c_\beta(\lambda) = \dfrac{2^{-2\lambda_\beta}\Gamma(2\lambda_\beta)}{\Gamma\left(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{1}{2}\right)\Gamma\left(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{m_\beta}{2}\right)}$ for $\beta \in \Sigma_*^+$

$c(\lambda) = c_{\mathrm{HC}} \prod_{\beta \in \Sigma_*^+} c_\beta(\lambda)$

where c_{HC} is a normalizing constant so that $c(\rho) = 1$.

Many rules: e.g. if both β and $\beta/2$ are roots, then $m_{\beta/2}$ is even and m_β is odd.

Examples

$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$: $\Sigma_*^+ = \Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

$[c(i\lambda)c(-i\lambda)]^{-1} \asymp \prod_{\alpha \in \Sigma^+} \lambda_\alpha \tanh(\pi\lambda_\alpha)$

G/K of even multiplicities (i.e. $\Sigma_*^+ = \Sigma^+$ and $m_\beta \in 2\mathbb{N}$ for all $\beta \in \Sigma^+$)

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$[c(i\lambda)c(-i\lambda)]^{-1}$ is a polynomial

$\widetilde{\rho}_\beta = \frac{1}{2}\Big(\frac{m_{\beta/2}}{2} + m_\beta\Big)$

Lemma

$\Pi(\lambda) = \prod_{\beta\in\Sigma_*^+} \lambda_\beta$,

$P(\lambda) = \prod_{\beta\in\Sigma_*^+} \Big(\prod_{k=0}^{(m_{\beta/2})/2-1} \big[i\lambda_\beta - \big(\frac{m_{\beta/2}}{4} - \frac{1}{2}\big) + k\big] \prod_{k=0}^{2\widetilde{\rho}_\beta-2}[i\lambda_\beta - (\widetilde{\rho}_\beta - 1) + k]\Big)$,

$Q(\lambda) = \prod_{\substack{\beta\in\Sigma_*^+ \\ m_\beta \text{ odd}}} \coth(\pi(\lambda_\beta - \widetilde{\rho}_\beta))$.

Then:

$$[c(\lambda)c(-\lambda)]^{-1} \asymp \Pi(\lambda)P(\lambda)Q(\lambda)$$

(empty products are equal to 1).

Hence: $[c(i\lambda)c(-i\lambda)]^{-1}$ *has at most first order singularities along the hyperplanes*

$$\mathcal{H}_{\beta,k,\pm} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_\beta = \pm i(\widetilde{\rho}_\beta + k)\}$$

where $\beta \in \Sigma_*^+$ *has multiplicity* m_β *odd and* $k \in \mathbb{Z}^+$.

Corollary

Set $L = \min\{\widetilde{\rho}_\beta|\beta| : \beta \in \Sigma_+^*, m_\beta$ *odd*$\}$.

Then, for every fixed $\omega \in \mathfrak{a}^*$ *with* $|\omega| = 1$*, the function* $r \mapsto [c(ir\omega)c(-ir\omega)]^{-1}$ *is holomorphic on* $\mathbb{C} \setminus i(]-\infty, -L] \cup [L, +\infty[)$.

Remark: $L = +\infty$ if m_β even for all $\beta \in \Sigma_+^*$.

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Extension of the resolvent of the Laplacian on $\mathbb{R}^n$

For $f \in C_c^\infty(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$

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where $\widehat{f}$ = Fourier transform of f

(entire of exp. type and rapidly decreasing by Paley-Wiener theorem)

Wanted: meromorphic continuation of $[R(z)f](y)$ from $z \in \mathbb{C}^+$ across $\mathbb{R}$.

Idea (for $n \geq 2$): polar coordinates

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For $\mathbb{R}^n$: $[R(z)f](y) \asymp \int_0^{+\infty} \frac{1}{r^2-z^2} \, F(r)\, r\, dr$

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n odd: i.e. F odd

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$\rightsquigarrow$ holomorphic extension to $\mathbb{C}$ by "shifting" path of integration.

n even: i.e. F even $\quad \rightsquigarrow \quad$ $r = e^\tau, \quad \tau \mapsto F(e^\tau)$ $\; i\pi$-periodic

$z = e^\zeta \in \mathbb{C}^+ \;\leftrightsquigarrow\; \zeta \in \{0 < \operatorname{Im} w < \pi\}$

$$
[R(e^\zeta)f](y) \asymp \int_{-\infty}^{+\infty} \frac{F(e^\tau)e^{2\tau}}{e^{2\tau} - e^{2\zeta}}\, d\tau
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$\rightsquigarrow$ holomorphic extension to the strip by "shifting" path of integration; logarithmic singularities

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&= \int_{-\infty}^{+\infty} \frac{F(r)}{r-z}\, dr
\end{aligned}
$$

$\rightsquigarrow$ holomorphic extension to $\mathbb{C}$ by “shifting” path of integration.

n even: i.e. F even $\quad \rightsquigarrow \quad \begin{array}{l} r = e^{\tau}, \quad \tau \mapsto F(e^{\tau}) \quad i\pi\text{-periodic} \\ z = e^{\zeta} \in \mathbb{C}^+ \;\leftrightsquigarrow\; \zeta \in \{0 < \operatorname{Im} w < \pi\} \end{array}$

$$
[R(e^{\zeta})f](y) \asymp \int_{-\infty}^{+\infty} \frac{F(e^{\tau})e^{2\tau}}{e^{2\tau} - e^{2\zeta}}\, d\tau
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$\rightsquigarrow$ holomorphic extension to the strip by “shifting” path of integration; logarithmic singularities

Extension of the resolvent of Δ on $X = G/K$

Suppose (real rank of X) $= \dim \mathfrak{a}^* =: n \geq 2$.

Let $f \in C_c^\infty(X)$ and $y \in X$ be fixed.

Polar coordinates in $\mathfrak{a}^*$ give

$$[R(z)f](y) \asymp \int_{\mathfrak{a}^*} \frac{1}{\langle\lambda,\lambda\rangle - z^2} (f \times \varphi_{i\lambda})(y) \frac{d\lambda}{c(i\lambda)c(-i\lambda)} = \int_0^\infty \frac{1}{r^2 - z^2} F(r) r \, dr$$

where

$$F(r) = F_{f,y}(r) = r^{n-2} \int_{S^{n-1}} (f \times \varphi_{ir\omega})(y) \frac{d\sigma(\omega)}{c(ir\omega)c(-ir\omega)}$$

is of the form

$$r^{n-2} \cdot \text{even holo function in } r \in \mathbb{C} \setminus i(]-\infty, -L] \cup [L, +\infty[).$$

- The Riemann surface M' above $\mathbb{C} \setminus -i[L, +\infty[$, to which R extends, depends on the parity of F, i.e. the parity of n.
- The holo/mero extension of R from M' to a Riemann surface M above $\mathbb{C}$ is equivalent to that of F near $-i[L, +\infty[$.
- The extension $\widetilde{R}$ of R to M is holomorphic on M' (because F is holo there). The poles of $\widetilde{R}$ on M (i.e. the resonances), if any, are precisely the poles of the extension of F to M. They are on the curve in M above $-i[L, +\infty[$.

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e.g.: if all root multiplicities are even (i.e. holo Plancherel density and $L = +\infty$), then M' is Riemann surface above $\mathbb{C}$ and get holo extension of R to $M = M'$.

Construction of the Riemann surface M' (no resonances there):

Theorem (Strohmaier, Mazzeo-Vasy, Hilgert-P.)

Let $f \in C_c^\infty(X)$ and $y \in X$ be fixed.

- *If the real rank n of X is odd: then $z \mapsto [R(z)f](y)$ is holomorphic in $z \in \mathbb{C}^+ := \{w \in \mathbb{C} : \mathrm{Im}\, w > 0\}$ and has holomorphic continuation to $\mathbb{C} \setminus \big(-i[L, +\infty[\big)$.*
- *If the real rank n of X is even: Let $\log$ denote the holomorphic branch of the logarithm defined on $\mathbb{C} \setminus]-\infty, 0]$ by $\log 1 = 0$. Set $\zeta = \log z$ for $z \in \mathbb{C}^+$ and set*

$$[R_{\log}(\zeta)f](y) = [R(e^\zeta)f](y) = \int_{-\infty}^{+\infty} \frac{1}{e^{2\tau} - e^{2\zeta}} F(e^\tau)e^{2\tau}\, d\tau.$$

Then the function $\zeta \mapsto [R_{\log}(\zeta)f](y)$ is holomorphic in $\zeta \in S_{0,\pi} := \{w \in \mathbb{C} : 0 < \mathrm{Im}\, w < \pi\}$ and has holomorphic extension to $\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}\setminus\{0\}} \Big(i\pi(n + \tfrac{1}{2}) + [\log L, +\infty[\Big)$.

The extended function satisfies: $[R_{\log}(\zeta + i\pi)f](y) = [R_{\log}(\zeta)f](y) + \pi i F(e^\zeta)$

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The rank 2 case

From the above, for every fixed $f \in C_c^\infty(X)$ and $y \in X$:

- The resolvent $z \in \mathbb{C}^+ \mapsto [R(z)f](y)$ extends holo from $\mathbb{C}^+$ to a Riemann surface M' (logarithmic cover) of $\mathbb{C} \setminus -i[L, +\infty[$.

Recall:
$L = \min\{\widetilde{\rho}_\beta |\beta| : \beta \in \Sigma_+^*, \, m_\beta \text{ odd}\}$.
$\widetilde{\rho}_\beta = \frac{1}{2}\left(\frac{m_{\beta/2}}{2} + m_\beta\right)$

- The possible poles of $\widetilde{R}$ (i.e. the possible resonances) are located above $-i[L, +\infty[$.
- The meromorphic continuation of $z \to [R(z)f](y)$ across $-i[L, +\infty[$ to a Riemann surface M above $\mathbb{C}$ (and containing M') is equivalent to the meromorphic continuation to M of

$$z \longrightarrow F(z) = \int_{S^1} (f \times \varphi_{iz\omega})(y) \, \frac{d\sigma(\omega)}{c(iz\omega)c(-iz\omega)}$$

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◇ meromorphic continuation to suitable Riemann surfaces over $\mathbb{C}$

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◇ range of the residue operators realized by irr admissible K-spherical reps of G

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The cases BC_2 and C_2 (except $\mathrm{SO}_0(p,2)$ with $p>2$ odd)

The rank-two irreducible Riemannian symm. spaces G/K with root system Σ of type BC_2 or C_2, with multiplicities $(m_\mathrm{l}, m_\mathrm{m}, m_\mathrm{s})$:

G	$\mathrm{SU}(p,2)$ $(p>2)$	$\mathrm{SO}_0(p,2)$ $(p>2)$	$\mathrm{Sp}(p,2)$ $(p\geq 2)$	$\mathrm{SO}^*(10)$	$E_{6(-14)}$
K	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(2))$	$\mathrm{SO}(p)\times\mathrm{SO}(2)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(2)$	$\mathrm{U}(5)$	$\mathrm{Spin}(10)\times\mathrm{U}(1)$
Σ	BC_2	C_2	$p=2$: C_2 $p>2$: BC_2	BC_2	BC_2
$(m_\mathrm{l}, m_\mathrm{m}, m_\mathrm{s})$	$(1,2,2(p-2))$	$(1,p-2,0)$	$(3,4,4(p-2))$	$(1,4,4)$	$(1,6,8)$

The long roots are the only roots with odd multiplicities if $G\neq\mathrm{SO}_0(p,2)$ with $p>2$ odd $\rightsquigarrow$ for $G\neq\mathrm{SO}_0(p,2)$ with $p>2$ odd, the resonances can studied by reduction to a direct product of two rank-one symmetric spaces.

J. Hilgert, A.P. and T. Przebinda (arxiv:1511.00488):

- meromorphic continuation to suitable Riemann surfaces over $\mathbb{C}$
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K-spherical reps of G

$\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$

$L = \frac{1}{2}\rho_X$ where $\rho_X > 0$ and $[\rho_X^2, +\infty)$ is the spectrum of Δ.

Theorem

Let $N \in \mathbb{Z}_{\geq 0}$ and $\mathbb{C}_N^- = \{z \in \mathbb{C} : 0 > \operatorname{Im} z > -(N + \frac{3}{2})\rho_X\}$.

- *There exists a Riemann surface M_N (explicit) over $\mathbb{C}_N^-$ such that for all $f \in C_c^\infty(X)$ and $y \in X$ the resolvent $z \to [R(z)f](y)$ extends meromorphically to a neighborhood of the curve γ_N lifting the interval $-i\big(0, (N + \frac{3}{2})\rho_X\big)$ to M_N.*
- *The meromorphically extended resolvent has simple poles precisely at the points of M_N above $z^{(n)} = -i(n + \frac{1}{2})$ with $n = 0, 1, 2, \ldots, N$*
- *The residue operator of the meromorphically extended resolvent at a point above $z^{(n)}$ (with $n \leq N$) is independent of N and given by*

$$\mathrm{Res}_n R : f \in C_c^\infty(X) \to f \times \varphi_{(n+\frac{1}{2})\rho}$$

$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$: Residue operators

The range of the residue operator $\quad \mathrm{Res}_n R : f \in C_c^\infty(X) \to f \times \varphi_{(n+\frac{1}{2})\rho} \quad$ at a point above $z^{(n)}$ in terms of spherical representations of $G = \mathrm{SL}(3,\mathbb{R})$.

Eigenspace representations:

$\mathbb{D}(X)$ = commutative algebra of G-invariant differential operators on X

$\mathcal{E}_\lambda(X)$ = joint eigenspace of $\mathbb{D}(X)$ of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$
$= \{f \in C^\infty(X) : Df = \gamma(D)(\lambda)f \text{ for all } D \in \mathbb{D}(X)\}$.

where $\gamma : \mathbb{D}(X) \to S(\mathfrak{a}_{\mathbb{C}})^W$ is the Harish-Chandra homomorphism.
e.g. $\gamma(\Delta)(\lambda) = \langle\rho,\rho\rangle - \langle\lambda,\lambda\rangle$

$(\mathcal{E}_\lambda(X), T_\lambda)$ = eigenspace representation of G, where
$[T_\lambda(g)f](x) = f(g^{-1}x) \qquad (g \in G,\ f \in \mathcal{E}_\lambda(X),\ x \in X)$

Eigenspace representations:

$\mathrm{Res}_n R(C_c^\infty(X))$ is the closed subspace of $\mathcal{E}_{(n+\frac{1}{2})\rho}(X)$ generated by the G-translates of $\varphi_{(n+\frac{1}{2})\rho}$: invariant, irreducible, infinite dimensional.

Principal series representations:

$(\mathrm{Res}_n R(C_c^\infty(X))$, left regular rep) is infinitesimally equiv to the unique irreducible subquotient of the non-unitary spherical principal series

$$\mathrm{Ind}_{MAN}^G(1 \otimes e^{(n+\frac{1}{2})\rho} \otimes 1)$$

containing the trivial K-type. This subquotient is **infinite dim**. It is **unitary** iff $n = 0$.

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