Higher algebra over the Leibniz operad

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50th Seminar "Sophus Lie"

Outline*

1. The supergeometry of Loday algebroids (J. Geo. Mech., 2013)
2. Free Courant and derived Leibniz algebroids (J. Geo. Mech., 2016)
3. Infinity category of homotopy Leibniz algebras (Theo. Appl. Cat., 2014)
4. A tale of three homotopies (Appl. Cat. Struct., 2015)

*Joint with V. Dotsenko, J. Grabowski, B. Jubin, D. Khudaverdian, J. Qiu, K. Uchino

The supergeometry of Loday algebroids
(J. Geo. Mech., 2013)

Motivations

- Double of a Lie bialgebra $\mathfrak{g}$ is a Lie algebra: $\mathfrak{g} \oplus \mathfrak{g}^*$
- Double of a Lie bialgebroid is a Courant algebroid: $\mathrm{T}\,M \oplus \mathrm{T}^*\,M$, E
- Leibniz bracket – **derived brackets**
- $[X, fY] = f[X, Y] + \lambda(X)f\,Y \rightsquigarrow$ classical Leibniz algebroids: wrong!

Loday algebroids: first attempt

'Definition': A Loday algebroid is a Leibniz bracket $[-,-]$ on sections of a vb E together with a left and right anchor

- If $\mathrm{rk}(E) = 1$, $[-,-]$ is AS and 1st order
- If $\mathrm{rk}(E) > 1$, $[-,-]$ is 'locally' a LAD bracket

'No' new examples $\rightsquigarrow$ modify 'definition'

Loday algebroids: second attempt

$$[X, fY] = f[X,Y] + \lambda(X)f\ Y$$

$$[X^i e_i, fY^j e_j] = X^i a^k_{ij} f Y^j\ e_k + X^i \lambda^a_i \partial_a f\ Y^j e_j + X^i \lambda^a_i\ f\ \partial_a Y^j\ e_j - Y^i \lambda^a_i \partial_a X^j\ e_j$$

$$\lambda(X)(\mathrm{d}\, f \otimes Y) = X^i \lambda^{ak}_{ij} \partial_a f\ Y^j e_k$$

Derivation in f, $\mathcal{C}^\infty(M)$-linear in X and Y, valued in sections

$$\lambda : \Gamma(E) \overset{\mathcal{C}^\infty(M)-\text{lin}}{\longrightarrow} \Gamma(TM) \otimes_{\mathcal{C}^\infty(M)} \mathrm{End}_{\mathcal{C}^\infty(M)} \Gamma(E)$$

$\lambda : E \to TM \otimes \mathrm{End}\, E \rightsquigarrow$ generalized anchor

Cohomology theory $\rightsquigarrow$ traditional left anchor λ

Definition [Grabowski, Khudaverdian, P, '13]

Definition

A Loday algebroid (LodAD) is a Leibniz bracket on sections of a vb $E \to M$ together with two bundle maps $\lambda : E \to TM$ and $\rho : E \to TM \otimes \operatorname{End} E$ such that

$$[X, fY] = f[X, Y] + \lambda(X)f\, Y$$

and

$$[fX, Y] = f[X, Y] + \rho(Y)(\mathrm{d}\, f \otimes X).$$

Examples

- Leibniz algebra
- (twisted) Courant-Dorfman ($\mathrm{T}\,M \oplus \mathrm{T}^*\,M$)
- Grassmann-Dorfman ($\mathrm{T}\,M \oplus \wedge\mathrm{T}^*\,M$ or $E \oplus \wedge E^*$)
- classical Leibniz algebroid associated to a Nambu-Poisson structure
- Courant algebroid
- ...

Courant: $f \in \mathcal{C}^\infty(M)\,,\, X, Y \in \Gamma(E)$

$D \in \mathrm{Der}(\mathcal{C}^\infty(M), \Gamma(E))$: $(Df|Y) = \frac{1}{2}\lambda(Y)f$

$D(fX|Y) = [fX, Y] + [Y, fX] \rightsquigarrow \rho(Y)(\mathrm{d}\,f \otimes X) = D(f)(X|Y)$

Derivation in f, $\mathcal{C}^\infty(M)$-linear in X and Y, valued in sections

Supergeometric interpretation

$$(E,[-,-],\lambda) \rightleftharpoons (\Gamma(\wedge E^*),\mathrm{d}) \rightleftharpoons \boxed{\mathrm{d} \in \mathrm{Der}_1(\Gamma(\wedge E^*),\wedge), \mathrm{d}^2 = 0}$$

Lie algebroids $\rightleftharpoons$ homological vfs on supermfds

Loday algebroids $\rightleftharpoons$?

Lie operator restricted to $\wedge_{\mathcal{C}^\infty(M)}(\Gamma(E),\mathcal{C}^\infty(M)) = \Gamma(\wedge E^*)$

Leibniz operator restricted to

$$\mathrm{Lin}_{\mathcal{C}^\infty(M)}\, D(\Gamma(E),\mathcal{C}^\infty(M)) = \Gamma(\otimes E^*) =: D(E)$$

$$(D \pitchfork \Delta)(X_1,\ldots,X_{p+q})$$

$$= \sum_{\sigma\in\mathrm{sh}(p,q)} \mathrm{sign}(\sigma) D(X_{\sigma_1},\ldots,X_{\sigma_p})\, \Delta(X_{\sigma_{p+1}},\ldots,X_{\sigma_{p+q}})$$

Supergeometric interpretation

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LodADs as hom vfs [Grabowski, Khudaverdian, P, '13]

Theorem

There is a 1-to-1 correspondence between LodAD structures $(E, [-,-], \lambda, \rho)$ *and equivalence classes of homological vfs*

$$\boxed{\mathrm{d} \in \mathcal{D}er_1(\mathcal{D}(E), \pitchfork), \mathrm{d}^2 = 0}$$

of the supercommutative space $(\mathcal{D}(E), \pitchfork)$*.*

Cartan calculus

Free Courant and derived Leibniz algebroids (J. Geo. Mech., '16)

Koszul duality for operads

Ginzburg-Kapranov, '94:

P_∞-algebra on $V \rightleftharpoons \mathrm{d} \in \mathrm{Der}_1\left(\mathbf{F}_{P^!}(sV^*)\right), \mathrm{d}^2 = 0$

Example:

L_∞-algebra on $V \rightleftharpoons$ homological vf on the formal smfd V

Geometric extensions:

L_∞-algebroid $\rightleftharpoons$ homological vf on a $\mathbb{N}$-smfd (Bonavolontà, P, '12)

LAD $\rightleftharpoons$ homological vf on a smfd

LodAD $\rightleftharpoons$ homological vf on a supercommutative space

Derived brackets induced by the homological vf

Courant algebroid

Classical LeiAD $(E, [-,-], \lambda)$ with a scalar product $(-|-)$

Invariance relations:

$$\lambda(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z])$$

$$\lambda(X)(Y|Z) = (X|[Y,Z] + [Z,Y])$$

Compatibility condition:

$$([X,Y]|Z) + (Y|[X,Z]) = (X|[Y,Z] + [Z,Y])$$

$\Gamma(E)$: $\mathcal{C}^\infty(M)$-module, $\mathcal{C}^\infty(M)$: commutative $\mathbb{R}$-algebra, $\mathbb{R}$: field $\rightsquigarrow$

$\mathcal{E}$: $\mathcal{A}$-module, $\mathcal{A}$: commutative R-algebra; R: commutative ring

Free Courant algebroid

Free Courant AD over an anchored $\mathcal{A}$-module $(\mathcal{E}, \lambda)$?

Free Leibniz algebra over the R-module $\mathcal{E}$: $(\underline{\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}})$

Free LeiAD over $(\mathcal{E}, \lambda)$: $(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \underline{\mathcal{F}(\lambda)}) \rightsquigarrow (-|-)_{\text{USP}}$?

$(\mathcal{E}_0, [-,-]_0, \lambda_0, (-|-)_0)$, $f : \mathcal{E} \to \mathcal{E}_0$, $f_1 : \mathcal{F}(\mathcal{E}) \to \mathcal{E}_0$, $X, Y \in \mathcal{F}(\mathcal{E})$

'$(X|Y)_{\text{USP}} =$' $(f_1(X)|f_1(Y))_0 = f_2(X \odot Y)$

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Invariance $\rightsquigarrow \mathcal{Q}(\mathcal{F}(\mathcal{E}))$ must have $\mathcal{F}(\mathcal{E})$-actions $\underline{\mu^\ell}$ and $\underline{\mu^r}$

$$\boxed{(\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{F}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\text{USP}})}$$

Well-DefNess of μ^ℓ and μ^r on $\mathcal{Q}(\mathcal{F}(\mathcal{E})) \rightsquigarrow$ $\boxed{\text{2 SymConds on } [-,-]_{\text{ULB}}}$

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Symmetric Leibniz algebroid [Jubin, P, Uchino, '16]

Definition

A symmetric LeiAD is a classical LeiAD $(\mathcal{E}, [-,-], \lambda)$ *s.th.*

$$X \circ fY - (fX) \circ Y = 0 \,,$$

$$([fX, Y] - f[X, Y]) \circ Z + Y \circ ([fX, Z] - f[X, Z]) = 0 \,,$$

where $X \circ Y := [X, Y] + [Y, X]$.

Examples

- Leibniz algebra
- (twisted) Courant-Dorfman
- Grassmann-Dorfman
- Courant algebroid

A LeiAD associated to Nambu-Poisson structure is NOT a symmetric LeiAD !

The free symmetric LeiAD over an anchored module is NOT Loday !

Generalized Courant AD [Jubin, P, Uchino, '16]

$$\boxed{(\mathcal{S}\mathcal{F}(\mathcal{E}), [-,-]_{\text{ULB}}, \mathcal{F}(\lambda), \mathcal{Q}(\mathcal{S}\mathcal{F}(\mathcal{E})), \mu^\ell, \mu^r, (-|-)_{\text{USP}})}$$

Definition

Generalized Courant AD: $\boxed{(\mathcal{E}_1, [-,-], \lambda, \mathcal{E}_2, \mu^\ell, \mu^r, (-|-))}$

Invariance relations:

$$\mu^\ell(X)(Y|Z) = ([X,Y]|Z) + (Y|[X,Z])$$

$$-\mu^r(X)(Y|Z) = ([Y,Z] + [Z,Y]|X)$$

Compatibility condition:

$$([X,Y]|Z) + (Y|[X,Z]) = ([Y,Z] + [Z,Y]|X)$$

Non-degeneracy $\Rightarrow$ symmetry AND $(\mathcal{C}^\infty(M), \lambda, -\lambda) \Rightarrow (\mathcal{E}_2, \mu^\ell, \mu^r)$

Generalized Courant AD [Jubin, P, Uchino, '16]

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Application

Question:

Which LeiAD brackets can be represented by a derived bracket ?

Answer:

Any symmetric LeiAD bracket has a universal derived bracket representation .

Application

Question:

Which LeiAD brackets can be represented by a derived bracket ?

Answer:

Any symmetric LeiAD bracket has a universal derived bracket representation .

Summary

Classical Leibniz AD

Loday AD

Symmetric Leibniz AD

Generalized Courant AD

Free generalized Courant AD

Universal derived bracket representation

Infinity category of homotopy Leibniz algebras (Theo. Appl. Cat., '14)

∞-Homotopies

$$\mathrm{Hom}_{P_\infty}(V,W) \simeq \mathrm{Hom}_{\mathtt{DGPiC}}(\mathcal{F}_{P\mathrm{i}}(V),\mathcal{F}_{P\mathrm{i}}(W))$$

$$\simeq \mathrm{MC}(\mathrm{Hom}_{\mathbb{R}}(\mathcal{F}_{P\mathrm{i}}(V),W)) =: \mathrm{MC}(\mathcal{C}) \quad (s^{-1} \text{ omitted})$$

Quillen Ho of $\mathrm{MC}(\mathcal{C})$: $\mathrm{MC}(\mathcal{C}\otimes\Omega_1)$

Gauge Ho of $\mathrm{MC}(\mathcal{C})$: "IC of specific VF"

P_∞-Ho: $\mathrm{MC}(\mathcal{C}\otimes\Omega_1)$

∞-Cat of P_∞-Alg [Khudaverdian, P, Qiu, '14]

P_∞-Ho = P_∞-2-Mor : MC($\mathcal{C} \otimes \Omega_1$)

P_∞-Mor = P_∞-1-Mor : MC($\mathcal{C} \otimes \Omega_0$)

Definition

$$P_\infty\text{-}(n+1)\text{-}\mathrm{Mor}\colon \mathrm{MC}(\mathcal{C} \otimes \Omega_n),\ n \geq 0$$

Getzler, '09:

$$\int \mathcal{C} \xrightarrow{\sim} \mathrm{MC}(\mathcal{C} \otimes \Omega_\bullet) : \infty\text{-GD}$$

Theorem

P_∞-`Alg`: *∞-Cat*

Application [Khudaverdian, P, Qiu, '14]

$\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$: ∞-Cat

$2\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$: 2-Cat (badly understood)

Theorem

∞-*Cat in* $\mathtt{Lei}_\infty\text{-}\mathtt{Alg} \twoheadrightarrow$ *2-Cat in* $2\mathtt{Lei}_\infty\text{-}\mathtt{Alg}$

Application of Getzler's integration technique

Answers to Baez-Crans and Schreiber-Stasheff

A tale of three homotopies
(Appl. Cat. Struct., ’16)

Equivalence of all ∞-homotopies [Dotsenko, P, '15]

Homotopy transfer theorem for homotopy cooperads

Explicit recipe to write a definition of operadic homotopy (nested trees)

Theorem

Concordances, Quillen, gauge, cylinder, and operadic homotopies are $\simeq$

THANK YOU