Short SL_3-structures on Lie algebras

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Freudenthal's magic square

A_1	A_2	C_3	F_4
A_2	$A_2 + A_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8

Let G be a centerless simple complex Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$.

Definition

A **short $\mathfrak{sl}_3$-structure** on $\mathfrak{g}$ is a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ isomorphic to $\mathfrak{sl}_3$ such that all irreducible components of its adjoint representation in $\mathfrak{g}/\mathfrak{l}$ are three- or one-dimensional.

The adjoint representation of $\mathfrak{l}$ in $\mathfrak{g}$ defines an action of the corresponding subgroup $L \subset G$ (isomorphic to SL_3) by automorphisms of $\mathfrak{g}$.

Set

$Z(\mathfrak{l})$: the centralizer of $\mathfrak{l}$ in G,
$\mathfrak{z}(\mathfrak{l}) = \mathrm{Lie}(Z(\mathfrak{l}))$: the centralizer of $\mathfrak{l}$ in $\mathfrak{g}$,
$U = \mathbb{C}^3$: the space of the tautological representation of $\mathfrak{l} = \mathfrak{sl}_3$,
U^*: the dual space.

The isotypic decomposition of $\mathfrak{g}$ w.r.t. $\mathfrak{l}$ is $Z(\mathfrak{l})$-invariant and has the form

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{z}(\mathfrak{l}) \oplus (U \otimes V) \oplus (U^* \otimes V^*),$$

where V and V^* are dual representation spaces of $Z(\mathfrak{l})$, on which $\mathfrak{l}$ acts trivially. The subspaces

$$\mathfrak{g}_0 = \mathfrak{l} \oplus \mathfrak{z}(\mathfrak{l}), \quad \mathfrak{g}_1 = U \otimes V, \quad \mathfrak{g}_{-1} = U^* \otimes V^*$$

are the eigenspaces of the automorphism of order 3 of $\mathfrak{g}$ defined by a central element of L. They constitute a $\mathbb{Z}_3$-grading of $\mathfrak{g}$.

The invariance of the operation in $\mathfrak{g}$ under the group L implies that the commutators of the elements of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are given by the following formulas (where elements of U^* and V^* are marked with primes, and broken brackets denote the pairing):

$$[u \otimes v, u' \otimes v'] = (u \otimes u')_0 \langle v, v' \rangle + \langle u, u' \rangle \delta(v, v'),$$
$$[u_1 \otimes v_1, u_2 \otimes v_2] = (u_1 \times u_2) \otimes \nu(v_1, v_2),$$
$$[u_1' \otimes v_1', u_2' \otimes v_2'] = (u_1' \times u_2') \otimes \nu'(v_1', v_2'),$$

where

$(u \otimes u')_0$: the projection of the linear operator $u \otimes u'$ in U to $\mathfrak{l} = \mathfrak{sl}(U)$,
$\delta : V \times V' \to \mathfrak{z}(\mathfrak{l})$: some $Z(\mathfrak{l})$-equivariant bilinear map,
$u_1 \times u_2$: the "vector product", the element of U^* defined by

$$\langle u_1 \times u_2, u_3 \rangle = \det(u_1, u_2, u_3) \quad \text{for any} \quad u_3 \in U,$$

$\nu : V \times V \to V^*$ and $\nu' : V^* \times V^* \to V$: some symmetric $Z(\mathfrak{l})$-equivariant bilinear maps.

We will write $v_1 \circ v_2$ instead of $\nu(v_1, v_2)$ and v^2 instead of $v \circ v$, and similarly for ν'.

The Jacobi identity for three elements of $\mathfrak{g}_1$ means that the trilinear form

$$F(v_1, v_2, v_3) = \langle v_1 \circ v_2, v_3 \rangle \quad (v_1, v_2, v_3 \in V)$$

is symmetric and

$$\delta(v, v^2) = 0 \quad \text{for any} \quad v \in V.$$

Similarly, the Jacobi identity for three elements of $\mathfrak{g}_{-1}$ means that the trilinear form

$$F'(v_1', v_2', v_3') = \langle v_1' \circ v_2', v_3' \rangle \quad (v_1', v_2', v_3' \in V^*)$$

is symmetric and

$$\delta(v'^2, v') = 0 \quad \text{for any} \quad v' \in V^*.$$

The cubic forms

$$N(v) = F(v, v, v), \quad N'(v') = F'(v', v', v'),$$

will be called the **norm** and the **dual norm** of $\mathfrak{g}$.

The Jacobi identity for two elements of $U \otimes V$ and one element of $U^* \otimes V^*$ gives

$$\delta(v, v')x = v' \circ (v \circ x) - \langle v', x \rangle v + \frac{1}{3}\langle v, v' \rangle x$$

for $x \in V$. Similarly,

$$\delta(v, v')x' = v \circ (v' \circ x') - \langle v, x' \rangle v' + \frac{1}{3}\langle v, v' \rangle x'$$

for $x' \in V^*$.

Theorem

1) The group $Z(\mathfrak{l})$ is reductive;
2) the representation of $\mathfrak{z}(\mathfrak{l})$ in V (and in V^*) is faithful;
3) the elements $\delta(v, v')$ $\quad (v \in V, v' \in V^*)$ span $\mathfrak{z}(\mathfrak{l})$ as a vector space.
4) there is an involution σ (in fact, Weyl involution) of $\mathfrak{g}$, leaving invariant $\mathfrak{l}$ (and hence $\mathfrak{g}_0$) and permuting $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$.

Corollary

The algebra $\mathfrak{z}(\mathfrak{l})$ together with its representation in V (and V^*) and the map δ are reconstructed from the norm by the (mentioned above) formula

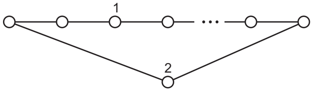
$$\delta(v, v')x = v' \circ (v \circ x) - \langle v', x \rangle v + \frac{1}{3}\langle v, v' \rangle x \quad \textit{for any} \quad x \in V.$$

All the short $\mathfrak{sl}_3$-structures on simple Lie algebras can be determined.

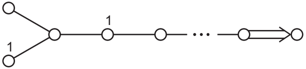
Theorem

Every simple Lie algebra $\mathfrak{g}$ but $C_n\,(n \geq 1)$ admits a short $\mathfrak{sl}_3$-structure, and such a structure is unique up to an inner automorphism of $\mathfrak{g}$. The Kac diagrams of the corresponding automorphisms of order 3 are given in the following table.

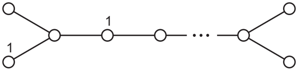
$A_n (n \geqslant 3)$



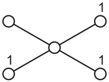
$B_n (n \geqslant 3)$



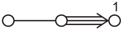
$D_n (n \geqslant 5)$



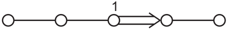
D_4



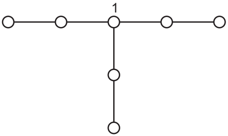
G_2



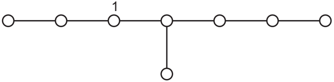
F_4



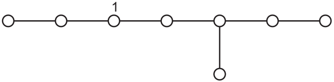
E_6



E_7



E_8



One can observe that in the case $\mathfrak{g} = A_n$, which we will call **degenerate**, the $\mathbb{Z}_3$-grading is "fake" in the sense that it is in fact a $\mathbb{Z}$-grading of depth one, considered as a $\mathbb{Z}_3$-grading, i.e $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$ or, equivalently, the form N is zero.

In all the other cases, which will be called **non-degenerate**, the form N is non-zero. In the following cases it is reducible:

if $\mathfrak{g} = G_2$ then $\dim V = 1$ and $N = x_1^3$;

if $\mathfrak{g} = B_3$ then $\dim V = 2$ and $N = x_1^2 x_2$;

if $\mathfrak{g} = D_4$ then $\dim V = 3$ and $N = x_1 x_2 x_3$;

if $\mathfrak{g} = B_n \quad (n \geq 4$ then $\dim V = 2n - 4$ and $N = (x_1^2 + \cdots + x_{2n-5}^2) x_{2n-4}$;

if $\mathfrak{g} = D_n \quad (n \geq 5$ then $\dim V = 2n - 5$ and $N = (x_1^2 + \cdots + x_{2n-6}^2) x_{2n-5}$.

In the cases $\mathfrak{g} = F_4, E_6, E_7, E_8$ the form N is irreducible and is the norm in a simple Jordan algebra of rank 3.

Choose vectors $u_0 \in U$ and $u_0' \in U^*$ so that $\langle u_0, u_0' \rangle = 1$ and denote by $\mathfrak{m}$ their common stabilizer in $\mathfrak{l}$. This is a subalgebra isomorphic to $\mathfrak{sl}_2$.

Definition

The centralizer $\mathfrak{h} = \mathfrak{z}(\mathfrak{m})$ of $\mathfrak{m}$ in $\mathfrak{g}$ is called the **frame** of $\mathfrak{g}$.

The frame is a reductive subalgebra intersecting each minimal $\mathfrak{l}$-invariant subspace of $\mathfrak{g}$ in a one-dimensional subspace. It depends on the choice of u_0 and u_0' and is defined up to a conjugation by L.

In particular, we have $\mathfrak{h} \cap \mathfrak{l} = \langle h \rangle$, where the element h can be normalized so that $hu_0 = 2u_0$ (and then $hu_0' = -2u_0'$). If we represent $\mathfrak{m} = \mathfrak{sl}_2$ as the left upper corner of $\mathfrak{l} = sl_3$, then $h = \mathrm{diag}(-1, -1, 2)$.

If we identify $u_0 \otimes V$ with V and $u_0' \otimes V^*$ with V^*, then

$$\mathfrak{h} = V^* \oplus (\langle h \rangle \oplus \mathfrak{z}(\mathfrak{l})) \oplus V.$$

This is a $\mathbb{Z}$-grading of depth one with the grading subspaces

$$\mathfrak{h}_{-1} = V^*, \quad \mathfrak{h}_0 = \langle h \rangle \oplus \mathfrak{z}(\mathfrak{l}), \quad \mathfrak{h}_1 = V,$$

which are the eigenspaces of $\mathrm{ad}(h)$ with eigenvalues $-2, 0, 2)$.

It is easy to see that

$$(u_0 \otimes u_0')_0 = \frac{1}{3} h.$$

It follows that the commutators of the elements of V with the elements of V^* are given by the formula

$$[v, v'] = \frac{1}{3} \langle v, v' \rangle h + \delta(v, v').$$

Let $H = Z(\mathfrak{m})$ be the centralizer of $\mathfrak{m}$ in G, so $\mathrm{Lie}(H) = \mathfrak{h}$ and H_0 be the centralizer of h in H. We have

$$H_0 = \mathbb{C}^* \cdot Z(\mathfrak{l})$$

(an almost direct product), where $\mathbb{C}^*$ is a one-dimensional torus with $\mathrm{Lie}(\mathbb{C}^*) = \langle h \rangle$.

It follows from the general theory of graded reductive Lie algebras that H_0 has an open orbit O in $\mathfrak{h}_1 = V$. There is the following alternative:

if the norm is zero (the degenerate case), then O is one $Z(\mathfrak{l})$-orbit,

if the norm is non-zero (the non-degenerate case), then O decomposes into one-parameter family of homothetic $Z(\mathfrak{l})$-orbits, the level varieties of the norm.

In the non-degenerate case, for any element $e \in O \subset \mathfrak{h}_1 = V$, there is a unique element $f \in \mathfrak{h}_{-1} = V^*$ such that (e, h, f) is an $\mathfrak{sl}_2$-triple, i.e.,

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The $\mathfrak{sl}_2$-triples arising in this way, are very special and can be classified a priori. They are characterized by the property that all irreducible components of their adjoint representation in $\mathfrak{h}$ are three- or one-dimensional.

One can show that the element f is proportional to $e^2 (= e \circ e)$; more precisely,

$$e^2 = \frac{1}{3} N(e) f.$$

Up to a constant factor, the norm of a simple Lie algebra $\mathfrak{g}$ is determined by the frame $\mathfrak{h}$ of $\mathfrak{g}$ as the only cubic form in $V = \mathfrak{h}_1$ invariant under $Z(\mathfrak{l})$. Thereby the algebra $\mathfrak{g}$ is reconstructed from its frame.

On the other hand, as we saw above, the frame (and thereby the very algebra $\mathfrak{g}$) can be reconstructed from the norm.

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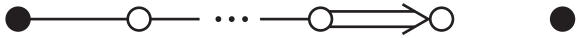
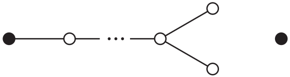
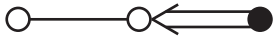
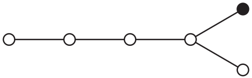
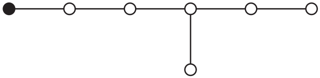
There is a simple characterization of cubic forms arising in this way.

Theorem

A cubic forms N in a vector space V is the norm of a simple Lie algebra if and only if the group of linear transformation preserving N is reductive and acts transitively on the variety $\{v \in V : N(v) = 1\}$.

In the following table, for each simple Lie algebra $\mathfrak{g}$ (other than C_n), we indicate its frame $\mathfrak{h}$ and the subalgebra $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{h}$. Besides, we represent the element h by its "numerical labels" on the Dynkin diagram of $\mathfrak{h}$. Namely, black (white) vertices correspond to the simple roots α with $\alpha(h) = 2$ ($\alpha(h) = 0$).

$\mathfrak{g}$	$\mathfrak{h} = \mathfrak{z}(\mathfrak{m})$	h	$\mathfrak{z}(\mathfrak{l})$
$A_n (n \geqslant 3)$	$A_{n-2} + T_1$		$A_{n-3} + T_1$
$B_n (n \geqslant 3)$	$B_{n-2} + A_1$		$B_{n-3} + T_1$
$D_n (n \geqslant 5)$	$D_{n-2} + A_1$		$D_{n-3} + T_1$
D_4	$A_1 + A_1 + A_1$		T_2
G_2	A_1		0
F_4	C_3		A_2
E_6	A_5		$A_2 + A_2$
E_7	D_6		A_5
E_8	E_7		E_6

Freudenthal's magic square

A_1	A_2	C_3	F_4
A_2	$A_2 + A_2$	A_5	E_6
C_3	A_5	D_6	E_7
F_4	E_6	E_7	E_8