Vector-valued orthogonal polynomials in several variables

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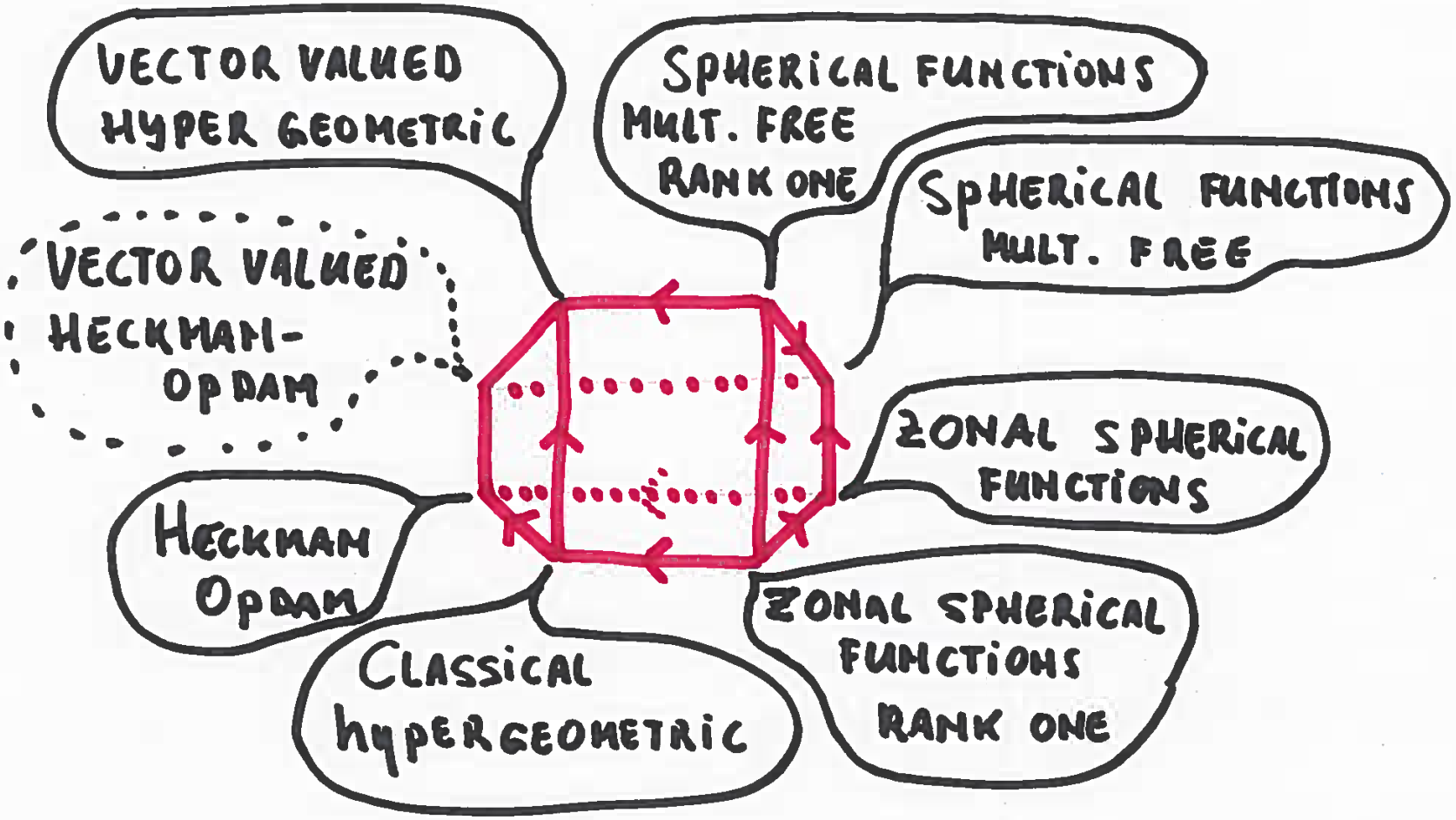

[ORTHOGONAL POLYNOMIALS]

Introduction

Goal

Understand collections of families of vector valued orthogonal polynomials

- *that are parametrized by real parameters,*
- *that are simultaneous eigenfunctions for a commutative algebra of differential operators,*
- *that allow for shift operators.*

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Approach:

- Collect an abundance of examples from representation theory
- Deform the examples

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Matrix valued orthogonal polynomials in one variable were studied already by KREIN *in the 1940s.*

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Matrix valued orthogonal polynomials in one variable were studied already by Krein *in the 1940s.*
Durán *(1997): Are there examples of MVOPs that are simultaneous eigenfunctions of a commutative algebra of differential operators?*
Yes! *Examples were found by* Grünbaum, Pacharoni *and* Tirao *(2002) using spherical functions for the pair* $(\mathrm{SU}(3), \mathrm{U}(2))$.

Introduction

Fix $r, N \in \mathbb{N}$ and denote $\mathbb{M} = \mathrm{End}(\mathbb{C}^N)$. Let $(\cdot)^*$ denote Hermitian adjoint.

- A **matrix valued polynomial** is an element $P \in \mathbb{M}[x]$.
- A **matrix weight** on an compact subset $I \subset \mathbb{R}^r$ is a map $W : I \to \mathbb{M}$ with $W(x)^* = W(x)$ and $W(x) > 0$ almost everywhere and

$$\int_I x^n W(x) dx \in \mathbb{M} \quad \text{(finite moments)}.$$

- Define the pairing

$$\langle P, Q \rangle_W = \int_I P(x)^* W(x) Q(x) dx \in \mathbb{M}$$

for $P, Q \in \mathbb{M}[x]$.

Introduction

The pairing $\langle \cdot, \cdot \rangle_W$ is a **matrix valued inner product**, i.e. it has the following properties:

- $\langle P, Q \rangle_W^* = \langle Q, P \rangle_W$,
- $\langle P, QA \rangle_W = \langle P, Q \rangle_W A$,
- $\langle P, P \rangle_W \geq 0$ and $\langle P, P \rangle_W = 0$ iff $P = 0$

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Definition

A family $\{P_d, d \in \mathbb{N}^r\} \subset \mathbb{M}[x]$ is called a family of matrix valued orthogonal polynomials, if (1) for all $n \in \mathbb{N}$, the set $\{P_d : |d| \leq n\}$ is a basis of the $\mathbb{M}$-submodule $\mathbb{M}[x]^n$, (2) for all $d, d' \in \mathbb{N}^r$, $\langle P_d, P_{d'} \rangle_W = \delta_{d,d'} C_d$ for some $C_d \in \mathbb{M}$ with $C_d > 0$.

Introduction

Fix a matrix weight W and a family of MVOPs $\{P_n : n \in \mathbb{N}\}$. Let $D \in \mathbb{M}[x] \otimes \mathbb{C}[\partial_x]$ and suppose that

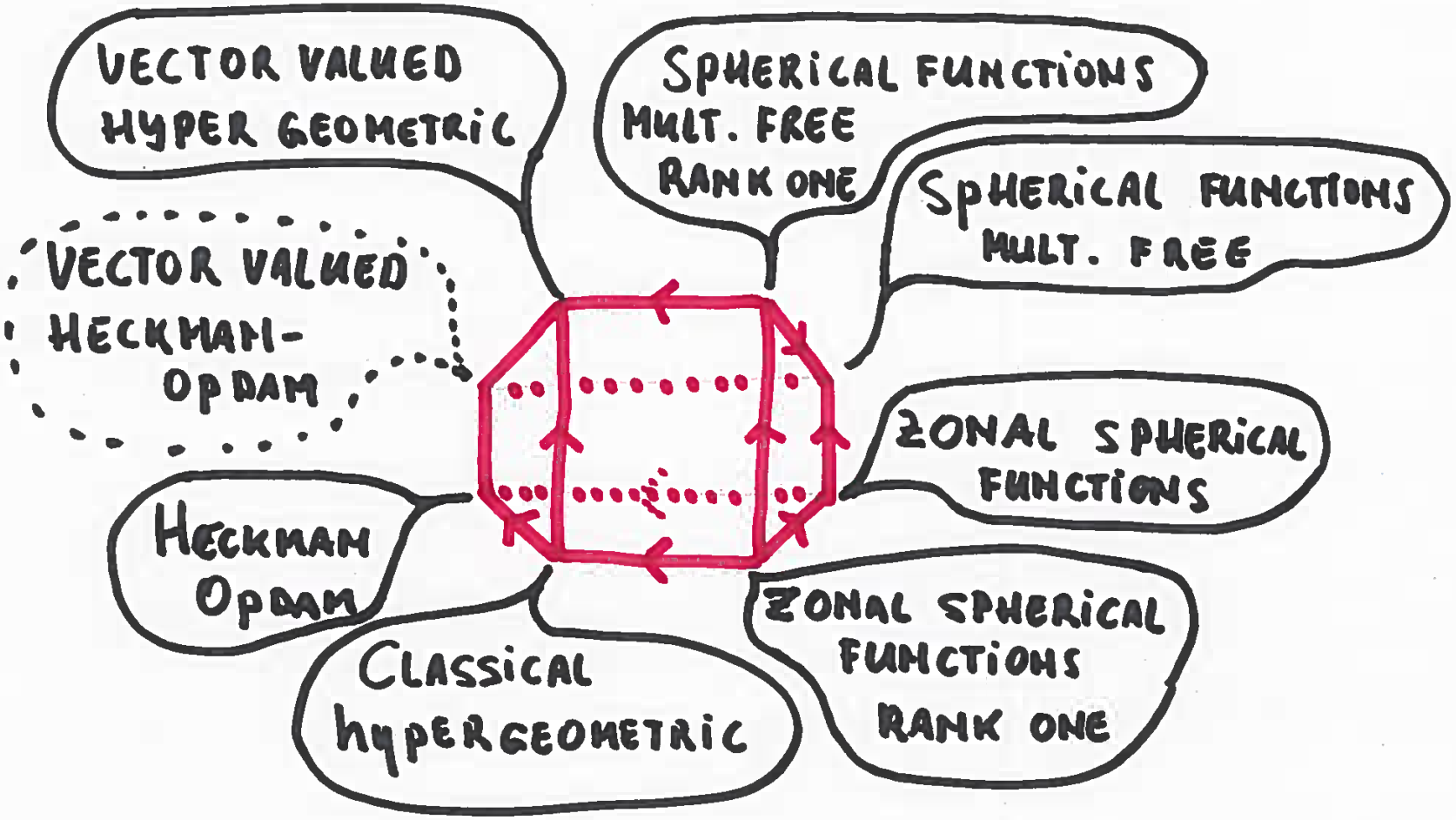
- $DP_n = P_n \Lambda_n$ for all n, $(\Lambda_n \in \mathbb{M})$,
- D is symmetric, i.e. $\langle DP, Q \rangle_W = \langle P, DQ \rangle_W$ for all $P, Q \in \mathbb{M}[x]$.

Then (W, D) is called a matrix valued classical pair (MVCP).

Introduction

Examples of families of MVOPs associated to spherical functions on compact symmetric spaces:

- $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diagSU}(2))$ by Koornwinder (1985), no differential operators are considered.
- $(\mathrm{SU}(3), \mathrm{U}(2))$ by Grünbaum, Pacharoni, Tirao (2002),
- $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diagSU}(2))$ revisited by Koelink, vP, Román (2012, 2013),
- $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ by Zurrian and Tirao (2013),
- ...



[ORTHOGONAL POLYNOMIALS]

Contents

- Group data
- Rank 1 examples
- Varying the parameter
- Examples of higher rank

Group data

Linear algebraic groups $/\mathbb{C}$.

- G connected, reductive group, $B \subset G$ a Borel subgroup
- $H \subset G$ a connected reductive subgroup,
- $P \subset H$ a parabolic subgroup such that G/P is *spherical*, i.e. admits an open B-orbit.
- (G, H, P) is called a multiplicity free system.

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Let $\mu : P \to \mathbb{C}^\times$ be a (positive) character. Then

- $(\pi_\mu^H, V_\mu^H) = \mathrm{ind}_P^H \mu$ is an irreducible H-representation.
- $\mathrm{ind}_H^G \pi_\mu^H$ is a multiplicity free G-representation.

Group data

- (vP, thesis 2012, Heckman & vP 2016) Classification of MFSs with G/H of rank one, uniform construction of families of MVOPs.
- (He, Nishiyama, Oichiai, Y. Oshima, 2013) Classification of MFSs with (G, H) a symmetric pair.
- (vP, preprint 2015) Classification of MFSs with (G, H) spherical, non-symmetric.

No.	G	H	H_*	J_H^c
1a	SL_{n+m}	$\mathrm{SL}_m \times \mathrm{SL}_n$ $m > n \geq 3$	$\mathrm{SL}_{m-n} \times S((\mathbb{C}^\times)^n)$	$\{\alpha_{m+1}\}, \{\alpha_{n+m-1}\}$
1b	SL_{n+2}	$\mathrm{SL}_n \times \mathrm{SL}_2$ $n \geq 3$ $m > n \geq 3$	$\mathrm{SL}_{n-2} \times S((\mathbb{C}^\times)^2)$	$\{\alpha_i\}$ $i = 2, \ldots, n-2, n+1$ unless $i = n-2 = 2$
1c	SL_{m+1}	$\mathrm{SL}_m, m \geq 2$	SL_{m-1}	$\Pi_H \backslash \{\alpha_i\}$
2	SL_{2n+1}	$\mathrm{Sp}_{2n} \times \mathbb{C}^\times$	$\mathbb{C}^\times$	$\emptyset$
3	SL_{2n+1}	Sp_{2n}	$\{e\}$	$\emptyset$
4	Sp_{2n}	$\mathrm{Sp}_{2n-2} \times \mathbb{C}^\times$	Sp_{2n-4}	$\forall \alpha_i, \alpha_j \in J^c$: $i < j < n \Rightarrow i, j-i \geq 3$
5	SO_{2n+1}	GL_n	$\{e\}$	$\emptyset$
6	SO_{4n+2}	SL_{2n+1}	$(\mathrm{SL}_2)^n$	$\{\alpha_1, \alpha_{2n}\}$
7	SO_{10}	$\mathrm{Spin}_7 \times \mathrm{SO}_2$	SL_2	$\emptyset$
8	SO_9	Spin_7	SL_3	$\{\alpha_1\}$
9	SO_8	G_2	SL_2	$\emptyset$
10	SO_7	G_2	SL_3	$\{\alpha_1\}, \{\alpha_2\}$
11	E_6	Spin_{10}	SL_4	$\emptyset$
12	G_2	SL_3	SL_2	$\{\alpha_1\}, \{\alpha_2\}$

Table: MFSs where (G, H) spherical, non-symmetric and G simple. For nos. 1a,b and c the roots of H, the semi-simple part of a maximal non-trivial Levi subgroup, are identified with the roots of G.

Group data

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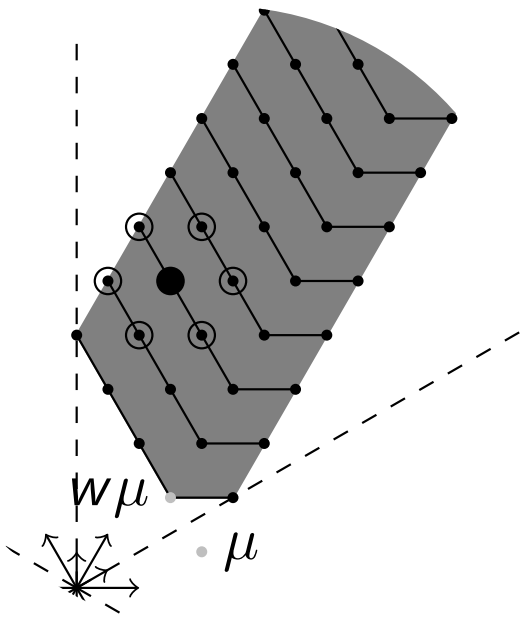

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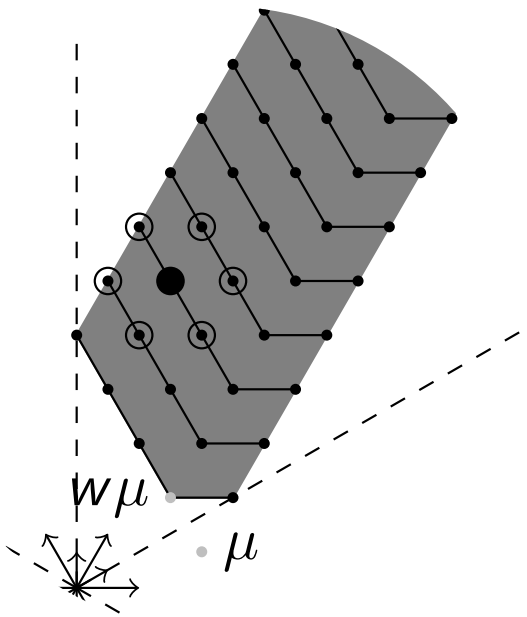

OK for rank one cases (checked by inverting the branching rules. Tough game for (F_4, B_4)).

Group data

- For several examples of higher rank this spectrum is known (vP, preprint 2015, Koelink & vP & Román, in progress).
- In progress (with Guido Pezzini): determining the spectra using algebraic geometry, spherical systems, combinatorics, calculating the *extended weight semi-group* after Roman Avdeev.

Rank 1 examples

Let (G, H, P) be a MFS with G/H symmetric of rank one and let $\mu : P \to \mathbb{C}^\times$ be a positive character. The spherical functions of type π_μ^H

- are the building blocks of $\left(\mathbb{C}[G] \otimes \mathrm{End}(V_\mu^H)\right)^{H \times H}$ (a module over the zonal spherical functions $\mathbb{C}[G]^{H \times H}$),
- are determined by their restriction to A, since they satisfy

$$F(h_1 g h_2) = \pi_\mu^H(h_1) F(g) \pi_\mu^H(h_2), \quad \text{for all } h_1, h_2 \in H \text{ and } g \in G.$$

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vP & Román, 2014: classification and explicit calculation of MVCPs obtained in this way, taking values in $\mathrm{End}(\mathbb{C}^2)$.
There are **parameters** around (the root mult. of underlying sym. space G/H) which can be varied continuously.

Varying the parameters

Theorem (Koelink, Martinez de los Riós, Román, 2014)

For $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diagSU}(2))$, *the*

- *family of MVOPs* $(P_n^{\mu,\nu} : n \in N)$,
- *algebra of differential operators* $\mathbb{D}(\mu, \nu)$,
- *orthogonality measure* $W^{\mu,\nu}(x)dx$

can be deformed continuously (similar to Gegenbauer) with $\nu > 0$. *Moreover,* $\partial P_n^{\mu,\nu} = nP_{n-1}^{\mu,\nu+1}$.

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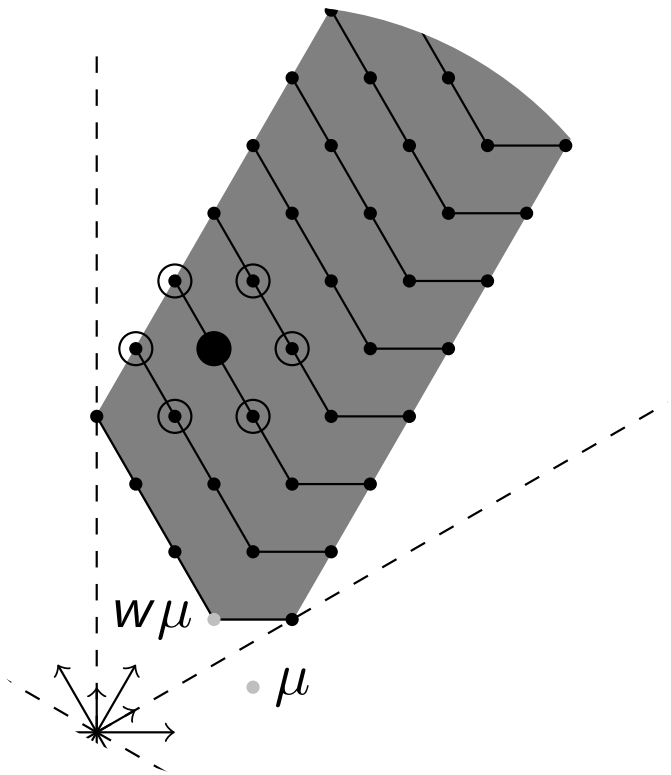
vP & Román, **preprint 2016**: Similar results for the examples of size 2×2, using different methods.This method also applies to the above case (with the same results).

Examples of higher rank

The spectrum of $\mathrm{ind}_H^G \pi_\mu^H$ provides the function $\Phi_0^\mu : A \to \mathrm{End}(\mathbb{C}^N)$, by arranging the restrictions to A of the spherical functions of degree zero in a matrix.



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$$\Phi_0^\mu(a) = D^\mu \cdot \pi_\mu(g_a) \cdot D^\mu.$$

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(a miraculous formula!)

Examples of higher rank

- Consider $(G, H) = (\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diagSU}(n+1))$, the MFSs correspond to representations $k\omega_1, k\omega_n$, with $k = 0, 1, \ldots$
- The decomposition of the G-module $\mathrm{ind}_H^G \pi_{k\omega_1}^H$ is known – the spectrum behaves well wrt tensor products.

We obtain MVOPs in several variables (taking values in $\mathrm{End}(\mathbb{C}^{d_k})$, with $d_k = \binom{n+k}{n}$).

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Note: the matrix U is hard to calculate. For the element g_a we have an explicit expression.

Examples of higher rank

Idea of proof:

- Candidate for g_a is $g_a = \Phi_0^{\mu_{\min}}(a)$, the 'minimal' spherical function at hand.
- The action of g_a on $V_{k\omega_1} = S^k(V_{\omega_1})$ can be calculated in two ways; comparing coefficients yields the result.
- The upper-triangular matrix has to do with specific equivariant embeddings in tensor products, whose decomposition yields terms of highest weights that are smaller than the one we started with – hence the matrix is upper triangular.
- The entries of the diagonal matrices D^μ in play are lengths of non-zero vectors. It follows that U is invertible.

Examples of higher rank

On $\mathrm{SU}(3) \times \mathrm{SU}(3)/\mathrm{diagSU}(3)$ we obtain

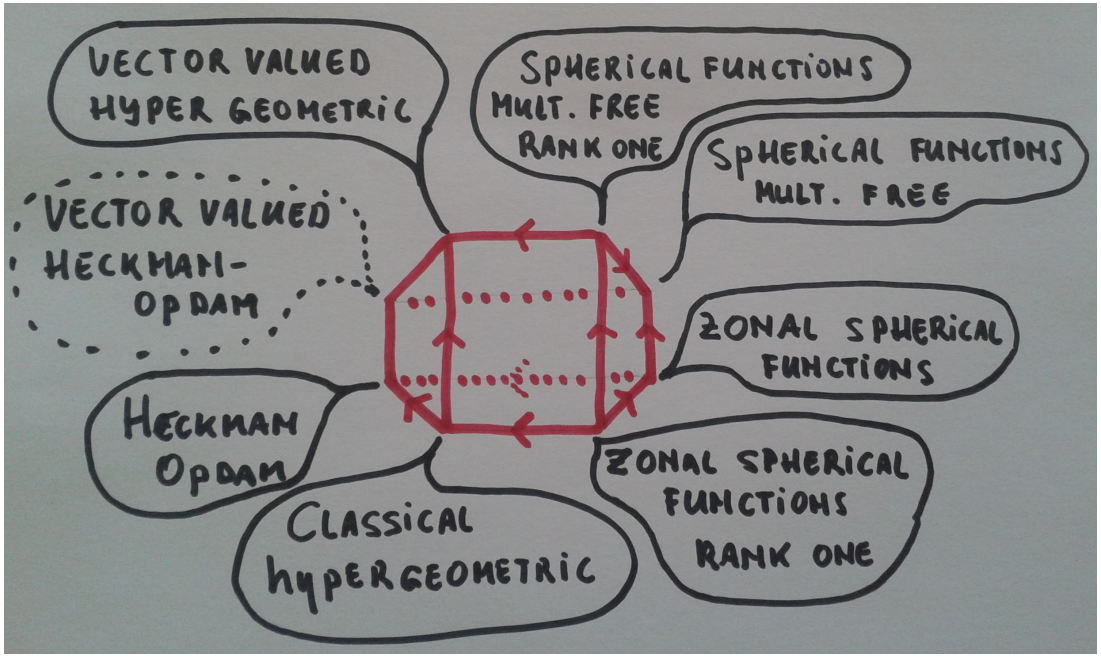
$$g_a = \begin{pmatrix} t_1 & \frac{1}{2}(t_3^{-1}t_2 + t_2^{-1}t_3) & t_1^{-1} \\ t_2 & \frac{1}{2}(t_1^{-1}t_3 + t_3^{-1}t_1) & t_2^{-1} \\ t_3 & \frac{1}{2}(t_2^{-1}t_1 + t_1^{-1}t_2) & t_3^{-1} \end{pmatrix}, \quad a = (t, t^{-1}), \quad t_1 t_2 t_3 = 1.$$

Outlook

- Collect the spectra of the multiplicity free G-modules $\mathrm{ind}_H^G \pi_\mu^H$ (with Pezzini).
- Write GAP-code to calculate the crucial functions Φ_0^μ for selected examples.
- Investigate deformations of higher rank examples (with Koelink and Román).
- Analogues for the non-compact duals of the compact symmetric spaces.

Challenges

- Understand $(U\mathfrak{g})^{\mathfrak{k}}/I(\mu)$ (a commutative quotient):
  - Harish-Chandra homomorphism,
  - Lepowsky map,
  - radial parts.
- Relate/connect to work of Dunkl, Griffeth: vector valued Jack polynomials.
- The spherical functions are Weyl group invariant. Are these results shadows of the representation theory of some Hecke algebra?
- ...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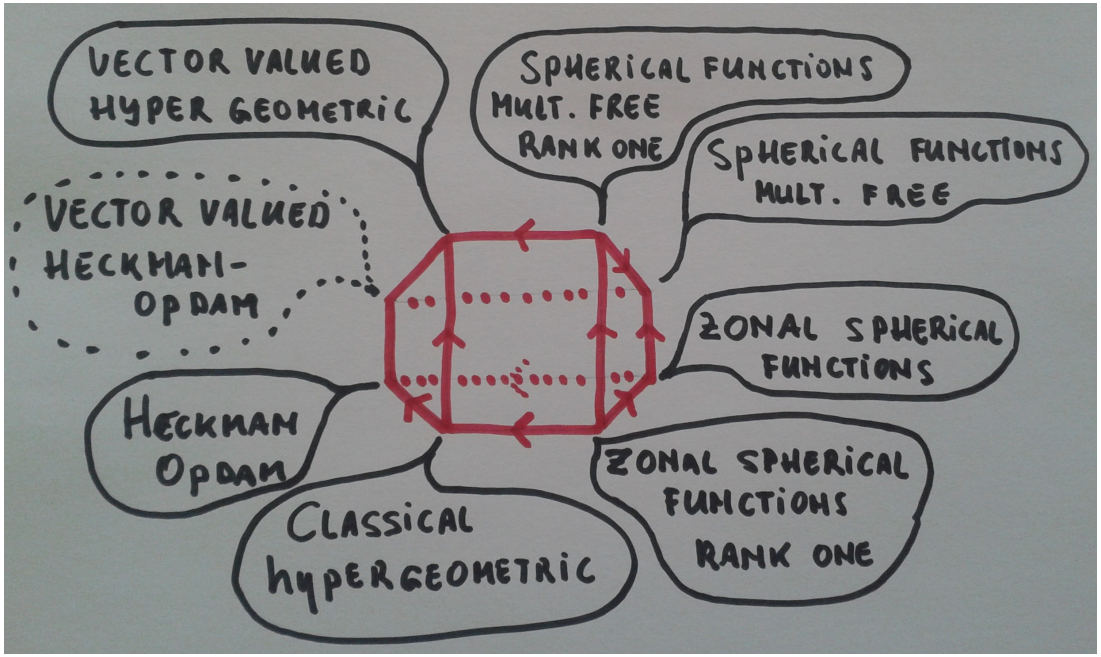
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Thank you!