Groupoids, coadjoint dynamical systems of solvable Lie groups, and their C^*-algebras

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References

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Overview

$\widehat{G} \ni [\pi] \rightsquigarrow \pi\colon C^*(G) \to \mathcal{B}(\mathcal{H}_\pi)$

- G type I $\implies \mathcal{K}(\mathcal{H}_\pi) \subseteq \boxed{\pi(C^*(G))} \simeq C^*(G)/\mathrm{Ker}\,\pi$

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New facts:

- G nilpotent Lie group, $\pi\colon G \to \mathcal{B}(\mathcal{X})$ irred., unif. bdd., $\mathcal{X}$ reflexive Banach sp. $\implies \mathcal{K}(\mathcal{X}) = \overline{\pi(\mathcal{C}_c(G))}^{\|\cdot\|}$
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- groupoids $\rightsquigarrow$ many examples of solvable Lie groups G (besides $\mathbb{C} \rtimes \mathbb{C}^*$) for which

$$(\exists[\pi] \in \widehat{G}) \quad \mathrm{Ker}\,\pi = \{0\} \text{ in } C^*(G)$$

Group C^*-algebras

- G locally compact group (e.g., Lie group)
- There is a Haar measure λ on G, invariant under left translations:
  $\int_G \varphi(a \cdot x) \mathrm{d}\lambda(x) = \int_G \varphi(x) \mathrm{d}\lambda(x)$ for $a \in G$ and $\varphi \in \mathcal{C}_c(G)$

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- *regular representation* $\lambda\colon \mathcal{C}_c(G) \to \mathcal{B}(L^2(G))$, $\lambda(f)\varphi = f * \varphi$,

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This is actually the *reduced* C^*-algebra of G.

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- $\widehat{G}$ = equivalence classes $[\pi]$ of unitary irreducible repres. $\pi\colon G\to U(\mathcal{H}_\pi)$
- $\widehat{\mathcal{A}}$ = equivalence classes of irreducible $*$-repres. of C^*-alg. $\mathcal{A}$
- *amenable group* means: canonical inclusion $\widehat{C^*(G)}\hookrightarrow\widehat{G}$ is bijective

Spectral topology

Recall: exponential Lie group $\Longrightarrow$ simply connected solvable Lie group $\Longrightarrow$ amenable group $\Longrightarrow$ $\widehat{G} \simeq \widehat{C^*(G)}$

Question

What topology can $\widehat{G}$ have if G is a simply connected solvable Lie group?

Def. C^*-algebra $\mathcal{A}$
$\leadsto$ spectrum $\widehat{\mathcal{A}} := \{[\pi] \mid \pi \colon \mathcal{A} \to B(\mathcal{H}_\pi)$ irred. $*$-repres.$\}$
$\leadsto$ topology with open sets $\{[\pi] \in \widehat{\mathcal{A}} \mid \pi|_{\mathcal{J}} \not\equiv 0\}$ for closed 2-sided ideals $\mathcal{J} \subseteq \mathcal{A}$

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- $\{[\pi_0]\}$ dense in $\widehat{\mathcal{A}}$ $\iff$ $\operatorname{Ker} \pi_0 = \{0\}$

Def. $\mathcal{A}$ is *primitive* if it has faithful irreducible representations

When is $C^*(G)$ primitive?

Old fact: G nilpotent Lie group
$\implies \pi(C^*(G)) = \mathcal{K}(\mathcal{H}_\pi)$ for all $[\pi] \in \widehat{G}$
$\implies C^*(G)$ is not primitive

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Proof. Let $[\pi] \in \widehat{G}$ with its coadjoint orbit $\mathcal{O}_\pi \subseteq \mathfrak{g}^*$.
- $\mathcal{O}_\pi$ is relatively open in its closure.
- $\{[\pi]\}$ dense in $\widehat{G} \simeq \mathfrak{g}^*/\mathrm{Ad}^*_G$

$\implies \mathcal{O}_\pi$ dense in $\mathfrak{g}^*$
$\implies \mathcal{O}_\pi$ open dense orbit
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- New fact: G has an even number of open coadjoint orbits

(e.g., the $ax+b$ group has 2 open orbits, nilpotent groups have none)

Q.E.D.

Connected Lie groups whose C^*-algebras are primitive

Theorem

Let $\tau\colon G \to \mathrm{End}\,(\mathcal{V})$ be a finite-dim. representation of a locally compact amenable group. Let $v_0 \in \mathcal{V}$ and $G(v_0) := \{g \in G \mid \tau(g)v_0 = v_0\}$.

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(i) $\pi \colon G \ltimes \mathcal{V}^*_{\mathbb{R}} \to \mathcal{B}(L^2(\mathcal{V}, \mu))$, $(\pi(g, \xi)\varphi)(v) := \mathrm{e}^{\mathrm{i}\xi(v)}\varphi(\tau(g^{-1})v)$ *is a continuous unitary irreducible representation.*

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Ex.: Let $\mathfrak{s} = \mathfrak{n}^- \dotplus \mathfrak{h} \dotplus \mathfrak{n}^+$ for a complex simple Lie alg. w.r.t. the Cartan subalgebra $\mathfrak{h}$. Define the Borel subalgebra $\mathfrak{g} := \mathfrak{h} \ltimes \mathfrak{n}^+$.

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Fact: G has a nonempty open coadjoint orbit
$\iff \mathfrak{s}\in\{\mathrm{so}(2\ell+1,\mathbb{C}),\mathfrak{sp}(2\ell,\mathbb{C}),\mathrm{so}(2\ell,\mathbb{C})$ with $\ell\in 2\mathbb{N}, E_7, E_8, F_4, G_2\}$
For such G, the above theorem applies with $\tau=\mathrm{Ad}_G^*$.

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For such G, the above theorem applies with $\tau = \mathrm{Ad}^*_G$. The simplest $G \ltimes \mathcal{V}^*_{\mathbb{R}}$ is 8-dimensional and is simply connected, solvable, of type I.

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transformation group C^*-algebra of $(G, \mathfrak{g}^*, \mathrm{Ad}^*_G)$

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Transformation groups are special cases of **groupoids**.

Groupoids as pullbacks of group bundles

- $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$ groupoid, with its domain/range maps $d, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$
- $x \in \mathcal{G}^{(0)} \rightsquigarrow$ its $\mathcal{G}$-orbit is $\mathcal{G}.x := \{r(g) \mid g \in \mathcal{G},\ d(g) = x\}$

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- define $\theta := (q|_\Sigma)^{-1} \circ q\colon \mathcal{G}^{(0)} \to \Sigma$ and the *pullback of* Π *by* θ

$$\theta^{\downarrow\downarrow}(\Pi) := \{(x, h, y) \in \mathcal{G}^{(0)} \times \Gamma \times \mathcal{G}^{(0)} \mid \theta(x) = \Pi(h) = \theta(y)\} \rightrightarrows \mathcal{G}^{(0)}$$

with projections on the 1st/3rd coordinates as domain/range maps

Groupoids as pullbacks of group bundles

- $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$ groupoid, with its domain/range maps $d, r\colon \mathcal{G} \to \mathcal{G}^{(0)}$
- $x \in \mathcal{G}^{(0)} \rightsquigarrow$ its $\mathcal{G}$-orbit is $\mathcal{G}.x := \{r(g) \mid g \in \mathcal{G},\ d(g) = x\}$

and isotropy group $\mathcal{G}(x) := \{g \in \mathcal{G} \mid d(g) = r(g) = x\}$

Example: group action $\mathcal{G} := G \times X \to X$, $d(g,x) = x$, $r(g,x) = g.x$

- the quotient map $q\colon \mathcal{G}^{(0)} \to \mathcal{G} \setminus \mathcal{G}^{(0)}$, $x \mapsto \mathcal{G}.x$
- select $\Sigma \subseteq \mathcal{G}^{(0)}$ that intersects every $\mathcal{G}$-orbit at exactly one point
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with projections on the 1st/3rd coordinates as domain/range maps

- Any map $\sigma\colon \mathcal{G}^{(0)} \to \mathcal{G}$ with $d \circ \sigma = \mathrm{id}$ defines a groupoid isomorphism

$$\Phi\colon \mathcal{G} \to \theta^{\downarrow\downarrow}(\Pi), \quad \Phi(g) := (r(g), \sigma(r(g))g\sigma(d(g))^{-1}, d(g))$$

Groupoid C^*-algebras

• a left Haar system of measures $\lambda = \{\lambda^x \text{ on } r^{-1}(x)\}_{x \in \mathcal{G}^{(0)}}$, satisfying
- continuity condition: $\mathcal{G}^{(0)} \ni x \mapsto \lambda(\varphi) := \int \varphi \mathrm{d}\lambda^x \in \mathbb{C}$ is cont.
- invariance condition: $\int \varphi(gh) \mathrm{d}\lambda^{d(g)}(h) = \int \varphi(h) \mathrm{d}\lambda^{r(g)}(h)$
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• regular repres. $\Lambda\colon \mathcal{C}_c(\mathcal{G}) \to \bigoplus_{x\in\mathcal{G}^{(0)}} \mathcal{B}(L^2(\mathcal{G}, \lambda^x))$, $\Lambda(\varphi)\psi_x := \varphi * \psi_x$

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- $\boxed{C^*(G) := \overline{\Lambda(\mathcal{C}(G))}^{\|\cdot\|}}$

As in the case of groups, this is actually the *reduced* C^*-algebra of G.

Morita equivalence of groupoid C^*-algebras

• $\theta\colon X \to Y$ has *local cross-sections* if for all $y \in Y$ and $x \in \theta^{-1}(y)$ there exist an open neighborhood V of y and $\tau\colon V \to X$ continuous with $\tau(y) = x$ and $\theta \circ \tau = \mathrm{id}_V$

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• **Recall:** for separable C^*-algebras,
$\mathcal{A}_1$ and $\mathcal{A}_2$ are Morita equivalent $\iff \mathcal{A}_1 \otimes \mathcal{K}(\mathcal{H}) \simeq \mathcal{A}_2 \otimes \mathcal{K}(\mathcal{H})$

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$\implies$ isomorphic lattices of ideals, representation theories etc.

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Def. A locally compact groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a *piecewise pullback of group bundles* with *pieces* V_k for $k = 1, \dots, n$ if

1. $V_k = U_k \setminus U_{k-1}$ for some open $\mathcal{G}$-invariant subsets

$$\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n = \mathcal{G}^{(0)}$$

2. $\mathcal{G}_{V_k} \simeq \theta_k^{\downarrow\downarrow}(\mathcal{T}_k)$ for an open continuous surjective map $\theta_k \colon V_k \to S_k$ having local cross-sections and a group bundle $\mathcal{T}_k \to S_k$

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There are closed 2-sided ideals $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = C^*(\mathcal{G})$ such that the $\mathcal{J}_k/\mathcal{J}_{k-1}$ is Morita equivalent to the C^*-algebra of sections of a continuous C^*-bundle whose fibers are C^*-algebras of isotropy groups of $\mathcal{G}$. Every isotropy group occurs for exactly one value of k.

Examples

Example 1 (nilpotent Lie group)
The Heisenberg group $\mathbb{H}_{2n+1}$ with $\mathfrak{h}_{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$,
$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(\langle x, y' \rangle - \langle y, x' \rangle))$
• $0 \to \mathcal{C}_0(\mathbb{R} \setminus \{0\}) \otimes \mathcal{K}(L^2(\mathbb{R}^n)) \to C^*(\mathbb{H}_{2n+1}) \to \mathcal{C}_0(\mathbb{R}^{2n}) \to 0$

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Groupoids with dense open orbits

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a locally compact groupoid, having a Haar system.

Proposition 1

Assume the orbits of $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ are locally closed. Then

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Proposition 2

If $U \subseteq \mathcal{G}^{(0)}$ is any open $\mathcal{G}$-invariant set and $x_0 \in U$, then one has:

(i) For every $x \in \mathcal{G}^{(0)} \setminus U$ the ideal $C^*(\mathcal{G}_U)$ of $C^*(\mathcal{G})$ is contained in the kernel of the regular representation $\Lambda_x \colon C^*(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}^x))$.

(ii) If U is an orbit of $\mathcal{G}$, then

$$\operatorname{Ker} \Lambda_{x_0} = \{0\} \iff \overline{U} = \mathcal{G}^{(0)}$$