Recent advances on Lie systems and their applications

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Abstract

After a quick presentation of the theory of Lie systems from a geometric perspective, recent progresses on their applications when compatible geometric structures exist will be described with an special emphasis in the particular case of admissible Kähler structures, and therefore with applications in Quantum Mechanics. The more general cases of quasi-Lie systems and bundle Lie systems will also be presented.

Outline

Lie–Scheffers systems: a quick review

Lie–Scheffers systems = Non-autonomous systems of first-order differential equations admitting a ...

Superposition rule: a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}, \quad x = \Phi(u_1, \dots, u_m; k_1, \dots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)(t)}; k_1, \dots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \dots, m\}$ being a generic set of particular solutions of the system and where $k_1, \dots, k_n$ are real numbers.

They are a generalisation of linear superposition rules for homogeneous linear systems for which $m = n$ and $x = \Phi(x_{(1)}, \dots, x_{(n)}; k_1, \dots, k_n) = k_1 \, x_{(1)} + \dots + k_n \, x_{(n)}$ but

i) The number m may be different from the dimension n.

ii) The function Φ is nonlinear in this more general case.

They appear quite often in many different branches of science ranging from pure mathematics to classical and quantum physics, control theory, economy, etc. Forgotten for a long time they had a revival due to the work of Winternitz and coworkers.

One particular example is Riccati equation, of a fundamental importance in physics (for instance factorisation of second order differential operators, Darboux transformations and in general Supersymmetry in Quantum Mechanics) and in mathematics.

These systems are related with equations in Lie groups and in general connections in fibre bundles.

In the solution of such non-autonomous systems of first-order differential equations we can use techniques imported form group theory, for instance Wei–Norman method, and reduction techniques coming from the theory of connections.

Recent generalisations have also been shown to be useful for dealing with other systems of differential equations (e.g. Emden–Fowler equations, Abel equations).

The existence of additional compatible geometric structures, like symplectic or Poisson structures may be useful in the search for solutions.

Theorem: *Given a non-autonomous system of n first order differential equations*

$$\frac{dx^i}{dt} = X^i(x^1, \ldots, x^n, t), \quad i = 1 \ldots, n,$$

a necessary and sufficient condition for the existence of a function $\Phi : \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$, $\quad x = \Phi(u_1, \ldots, u_m; k_1, \ldots, k_n)$, $u_a \in \mathbb{R}^n$, such that the general solution is

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)(t)}; k_1, \ldots, k_n) \ ,$$

with $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ being a set of particular solutions of the system and where $k_1, \ldots, k_n$, are n arbitrary constants, is that the system can be written as

$$\frac{dx^i}{dt} = b_1(t)\xi^i_1(x) + \cdots + b_r(t)\xi^i_r(x), \qquad i = 1 \ldots, n,$$

where $b_1, \ldots, b_r$, are r functions depending only on t and ξ^i_α, $\alpha = 1, \ldots, r$, are functions of $x = (x^1, \ldots, x^n)$, such that the r vector fields in $\mathbb{R}^n$ given by

$$X_\alpha \equiv \sum_{i=1}^{n} \xi^i_\alpha(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \ , \qquad \alpha = 1, \ldots, r,$$

close on a real finite-dimensional Lie algebra, i.e. the X_α are l.i. and there are r^3 real numbers, $c_{\alpha\beta}{}^\gamma$, such that

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^\gamma X_\gamma \ .$$

The number r satisfies $r \leq mn$.

The geometric concept of superposition rule is the following:

A superposition rule for a t-dependent vector field X in a n-dimensional manifold M is a map $\Phi : M^m \times M \to M$ such that if $\{x_{(1)}(t), \ldots, x_{(m)}(t)\}$ is a generic set of integral curves of X, then $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t), k)$, with $k \in M$ is also integral curve of X, and each integral curve is obtained in this way.

The result of the Theorem in modern terms is that a t-dependent vector field X admits a superposition rule if there exist r fields $X_1, \ldots, X_r$ in M and functions $b_1(t), \ldots, b_r(t)$ such that $X(x,t)$ be a linear combination

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t)\, X_\alpha(x).$$

The t-dependent vector field can be seen as a family of vector fields $\{X_t \mid t \in \mathbb{R}\}$.

Definition. *The minimal Lie algebra of a given a t-dependent vector field X on a manifold M is the smallest real Lie algebra, V^X, containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \mathrm{Lie}(\{X_t \mid t \in \mathbb{R}\})$.*

Definition. *The vector field associated to a non-autonomous system X allows us to define a generalised distribution $\mathcal{D}^X : x \in M \mapsto \mathcal{D}^X_x \subset TM$, where $\mathcal{D}_x = \{Y_x \mid Y \in V^X\} \subset T_xM$, and X also gives rise to a generalised co-distribution $\mathcal{V} : x \in M \mapsto \mathcal{V}_x \subset T^*M$, where $\mathcal{V}_x = \{\omega_x \mid \omega_x(Y_x) = 0, \forall Y_x \in \mathcal{D}^X_x\}$.*

Remark that the Lie–Scheffers theorem can be reformulated as follows:

Theorem: *A system X admits a superposition rule if and only if the minimal Lie algebra V^X is finite-dimensional.*

Definition. *A function $f : U \subset U^X \to \mathbb{R}$ is a local first integral (or t-independent constant of the motion) for a given t-dependent vector field X over $\mathbb{R}^n$ if $Xf = 0$*

Then f is a first integral if and only if $df \in \mathcal{V}^X|_U$.

One can easily prove that:

Property. *Given a t-dependent vector field X on a n-dimensional manifold M and a point $x \in U^X$ where the rank of $\mathcal{D}^X$ is equal to k, the associated co-distribution $\mathcal{V}^X$ admits, in a neighbourhood of x, a local basis of the form, $df_1, \ldots, df_{n-k}$, where, $f_1, \ldots, f_{n-k}$, is a family of fist integrals of X. Additionally, the space $\mathcal{I}^X|_U$ of first-integrals of the system X over an open U of M, can be put in the form*

$$\mathcal{I}^X|_U = \{g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{n-k} \to \mathbb{R},\ g = F(f_1, \ldots, f_{n-k})\}.$$

There exist different procedures to derive superposition rules for Lie systems. We can use a method based on the *diagonal prolongation* notion.

Definition. *Given a t-dependent vector field X over M, its diagonal prolongation to M^{m+1} is the t-dependent vector field $\widetilde{X}$ over M^{m+1} such that*

- ☐ $\widetilde{X}$ *projects onto X by the map $\mathrm{pr} : (x_{(0)}, \ldots, x_{(m)}) \in M^{m+1} \mapsto x_{(0)} \in M$, that is, $\mathrm{pr}_* \widetilde{X} = X$.*
- ☐ $\widetilde{X}$ *is invariant under permutation $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \ldots, m$.*

The procedure to determine superposition rules described is:

i) Take a basis $X_1, \ldots, X_r$ of the Vessiot–Guldberg Lie algebra V associated with the Lie system.

ii) Choose the minimum integer m such that the diagonal prolongations to M^m of the elements of the previous basis are linearly independent at a generic point.

ii) Obtain n common first-integrals for the diagonal prolongations, $\widetilde{X}_1, \ldots, \widetilde{X}_r$, to M^{m+1} (for instance, by means of *the method of characteristics*).

iii) Obtain the expression of the variables of one of the spaces M only in terms of the other variables of M^{m+1} and the above mentioned n first-integrals.

The so obtained expressions give rise to a superposition rule in terms of any generic family of m particular solutions and n constants corresponding to the possible values of the derived first-integrals.

Some particular examples

A) Inhomogeneous linear systems:

$$\frac{dx^i}{dt} = \sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \;, \qquad i = 1, \dots, n.$$

The time-dependent vector field is

$$X = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A^i{}_j(t)\, x^j + B^i(t) \right) \frac{\partial}{\partial x^i} \;,$$

which is a linear combination with t-dependent coefficients,

$$X = \sum_{i,j=1}^{n} A^i{}_j(t)\, Y_{ij} + \sum_{i=1}^{n} B^i(t)\, Y_i \;,$$

of the $n^2 + n$ vector fields

$$Y_{ij} = x^j \frac{\partial}{\partial x^i}, \qquad Y_i = \frac{\partial}{\partial x^i} \;, \qquad i, j = 1, \dots, n\,.$$

These last vector fields have the following commutation relations:

$$[Y_i, Y_k] = 0 \ , \qquad [Y_{ij}, Y_k] = -\delta_{kj}\, Y_i \ , \qquad \forall\, i, j, k = 1, \ldots, n\,.$$

• The set $\{Y_i \mid i = 1, \ldots, n\}$ generates an Abelian ideal.

• The set $\{Y_{ij} \mid i, j = 1, \ldots, n\}$ generates a Lie subalgebra.

• The Vessiot Lie algebra is isomorphic to the (n^2+n)-dimensional Lie algebra of the affine group.

In this case $r = n^2 + n$ and using the general theory one can see that $m = n + 1$ and the equality $r = m\,n$ also follows.

The superposition function $\Phi : \mathbb{R}^{n(n+1)} \to \mathbb{R}^n$ is:

$$x = \Phi(u_1, \ldots, u_{n+1}; k_1, \ldots, k_n) = u_1 + k_1(u_2 - u_1) + \cdots + k_n(u_{n+1} - u_1),$$

i.e. the general solution can be written in terms of $n+1$ generic solutions as:

$$\Phi(x_{(1)}, \ldots, x_{(n+1)}; k_1, \ldots, k_n) = x_{(n+1)} + k_1(x_{(1)} - x_{(n+1)}) + \cdots + k_n(x_{(n)} - x_{(n+1)}).$$

B) The Riccati equation ($n = 1$)

$$\frac{dx(t)}{dt} = a_2(t)\, x^2(t) + a_1(t)\, x(t) + a_0(t)\ .$$

Now $m = r = 3$ and the superposition principle comes from the relation

$$\frac{x - x_1}{x - x_2} : \frac{x_3 - x_1}{x_3 - x_2} = k\ ,$$

or in other words,

$$x(t) = \frac{x_1(t)(x_3(t) - x_2(t)) + k\, x_2(t)(x_1(t) - x_3(t))}{(x_3(t) - x_2(t)) + k\,(x_1(t) - x_3(t))}\ .$$

i.e. the superposition rule involves three different solutions, $m = 3$. The value $k = \infty$ must be accepted, otherwise we do not obtain the solution x_2.

The vector fields $Y^{(1)}$, $Y^{(2)}$ and $Y^{(3)}$ are given by

$$Y^{(1)} = \frac{\partial}{\partial x}\,, \quad Y^{(2)} = x\,\frac{\partial}{\partial x}\,, \quad Y^{(3)} = x^2\,\frac{\partial}{\partial x}\,,$$

that close on a three-dimensional real Lie algebra, i.e. $r = 3$, with defining relations

$$[Y^{(1)}, Y^{(2)}] = Y^{(1)}\,, \quad [Y^{(1)}, Y^{(3)}] = 2Y^{(2)}\,, \quad [Y^{(2)}, Y^{(3)}] = Y^{(3)}\,.$$

Then, the associated Lie algebra is $\mathfrak{sl}(2, \mathbb{R})$.

C) Lie–Scheffers systems on Lie groups

M is a Lie group G. Consider a basis of either left–invariant (or right–invariant) vector fields X_α in G as corresponding to the Lie algebra $\mathfrak{g}$ of G or its opposite algebra.

If $\{a_1, \ldots, a_r\}$ is a basis for the tangent space T_eG and X_α^R denotes the right-invariant vector field in G such that $X_\alpha^R(e) = a_\alpha$, a Lie–Scheffers system is

$$\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha^R(g(t)) \ .$$

When applying $(R_{g(t)^{-1}})_{*g(t)}$ to both sides we obtain the equation on T_eG

$$(R_{g(t)^{-1}})_{*g(t)}(\dot{g}(t)) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ , \qquad (**)$$

This is usually written with a slight abuse of notation:

$$(\dot{g}\, g^{-1})(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \ .$$

Such equation is right-invariant. Then,

If $\bar{g}(t)$ is a solution of (**) with initial condition $\bar{g}(0) = e$, the solution $g(t)$ with initial conditions $g(0) = g_0$ is given by $\bar{g}(t)g_0$.

Moreover, there is a superposition rule $\Phi : G \times G \to G$ involving one solution

$$\Phi(g, g_0) = g\, g_0.$$

This example is very useful because there are many other examples related with them as explained next.

The motivation for the choice of the minus sign on the right hand side will be clear shortly.

D) Lie-Scheffers systems on homogeneous spaces for Lie groups

Let H be a closed subgroup of G and consider the homogeneous space $M = G/H$. The right–invariant vector fields X^R_α are τ-projectable and the τ-related vector fields in M are the fundamental vector fields $-X_\alpha = -X_{a_\alpha}$ corresponding to the natural left action of G on M.

$$\tau_{*g} X^R_\alpha(g) = -X_\alpha(gH) \ ,$$

and we will have an associated Lie–Scheffers system on M:

$$X(x,t) = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha(x) \ .$$

Therefore, a solution of this last system starting from x_0 will be:

$$x(t) = \Phi(g(t), x_0) \ ,$$

with $g(t)$ being a solution of (**).

The converse property is true: Given a Lie Scheffers system defined by complete vector fields with associated Lie algebra $\mathfrak{g}$, we can see these as fundamental vector fields relative to an action which can be found by integrating the vector fields.

Wei-Norman method

Let G be a r–dimensional Lie group and $\{a_1, \ldots, a_r\}$ a basis of T_eG, consider the equation determining the curves $g(t) \in G$ such that

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \in T_eG \ ,$$

with $g(0) = e \in G$.

In order to solve directly such equation we can use a method which is a generalisation of the one proposed by Wei and Norman for finding the time evolution operator for a linear systems of type

$$\frac{dU(t)}{dt} = H(t)U(t) \ ,$$

with $U(0) = I$.

Remark that there exist alternative methods for solving the equation by reducing the problem to a simpler one.

Property: If $g(t)$, $g_1(t)$ and $g_2(t)$ are differentiable curves in G such that $g(t) = g_1(t)g_2(t)$, $\forall t \in \mathbb{R}$, then,

$$\begin{aligned} R_{g(t)^{-1}*g(t)}(\dot{g}(t)) &= R_{g_1(t)^{-1}*g_1(t)}(\dot{g}_1(t)) \\ &+ \mathrm{Ad}\,(g_1(t))\left\{R_{g_2(t)^{-1}*g_2(t)}(\dot{g}_2(t))\right\} \,. \end{aligned}$$

The generalisation to several factors is as follows:

If $g(t) = g_1(t)g_2(t)\cdots g_l(t) = \prod_{i=1}^{l} g_i(t)$, then we have:

$$R_{g(t)^{-1}*g(t)}(\dot{g}(t)) = \sum_{i=1}^{l}\left(\prod_{j<i}\mathrm{Ad}\,(g_j(t))\right)\left\{R_{g_i(t)^{-1}*g_i(t)}(\dot{g}_i(t))\right\} \,,$$

where it has been taken $g_0(t) = e$ for all t.

The generalized Wei–Norman method consists on writing $g(t)$ in terms of its second kind canonical coordinates,

$$g(t) = \prod_{\alpha=1}^{r}\exp(-v_\alpha(t)a_\alpha) = \exp(-v_1(t)a_1)\cdots\exp(-v_r(t)a_r) \,,$$

and transforming the equation into a differential equation system for the $v_\alpha(t)$, with initial conditions $v_\alpha(0) = 0$ for all $\alpha = 1, \ldots, r$.

Then, using the expression of the above property, with $l = r = \dim G$ and $g_\alpha(t) = \exp(-v_\alpha(t) a_\alpha)$ for all α, we see that

$$\begin{aligned} R_{g(t)^{-1}\,*g(t)}(\dot g(t)) &= -\sum_{\alpha=1}^{r} \dot v_\alpha \left(\prod_{\beta<\alpha} \mathrm{Ad}\,(\exp(-v_\beta(t) a_\beta))\right) a_\alpha \\ &= -\sum_{\alpha=1}^{r} \dot v_\alpha \left(\prod_{\beta<\alpha} \exp(-v_\beta(t) \mathrm{Ad}\,(a_\beta))\right) a_\alpha \,. \end{aligned}$$

Then, the fundamental expression of the Wei–Norman method is:

$$\sum_{\alpha=1}^{r} \dot v_\alpha \left(\prod_{\beta<\alpha} \exp(-v_\beta(t) \mathrm{Ad}\,(a_\beta))\right) a_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha \,,$$

with $v_\alpha(0) = 0$, $\alpha = 1, \ldots, r$.

The resulting system of differential equations for the functions $v_\alpha(t)$ is integrable by quadratures if the Lie algebra is solvable, and in particular, for nilpotent Lie algebras.

The reduction method

Given an equation on a Lie group

$$\dot{g}(t)\, g(t)^{-1} = a(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\, a_\alpha \in T_eG\ , \qquad (\bullet)$$

with $g(0) = e \in G$, it may happen that the only nonvanishing coefficients are those corresponding to a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the equation reduces to a simpler equation on a subgroup, involving less coordinates.

The fundamental result is that if we know a particular solution of the problem associated in a homogeneous space, the original solution reduces to one on the subgroup.

Let us choose a curve $g'(t)$ in the group G, and define the curve $\overline{g}(t)$ by $\overline{g}(t) = g'(t)g(t)$. The new curve in G, $\overline{g}(t)$, determines a new Lie system.

Indeed,

$$R_{\overline{g}(t)^{-1}*\overline{g}(t)}(\dot{\overline{g}}(t)) = R_{g'^{-1}(t)*g'(t)}(\dot{g}'(t)) - \sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{Ad}\,(g'(t))a_\alpha \ ,$$

which is an equation similar to the original one but with a different right hand side.

In this way we can define an action of the group of curves in the Lie group G on the set of Lie systems on the group. This can be used to reduce a given Lie system to a simpler one.

The aim is to choose the curve $g'(t)$ in such a way that the new equation be simpler. For instance, we can choose a subgroup H and look for a choice of $g'(t)$ such that the right hand side lies in T_eH, and hence $\overline{g}(t) \in H$ for all t.

If $\Psi : G \times M \to M$ is a transitive action of G on a homogeneous space M, which can be identified with the set G/H of left-cosets, by choosing a fixed point x_0, then the integral curves starting from the point x_0 associated to both Lie systems are related by

$$\overline{x}(t) = \Psi(\overline{g}(t), x_0) = \Psi(g'(t)g(t), x_0) = \Psi(g'(t), x(t)) \ .$$

Therefore, this gives an action of the group of curves in G on the set of associated Lie systems in homogeneous space s.

More explicitly, if we consider ta curve $g'(t)$ in the group, the Lie system transforms into a new one

$$\dot{\bar{x}} = \sum_{\alpha=1}^{r} \bar{b}_\alpha(t) X_\alpha(\bar{x}) \ ,$$

in which

$$\bar{b} = \mathrm{Ad}\,(g'(t))b(t) + \dot{g}'\,g'^{-1} \ .$$

The important result is that the knowledge of a particular solution of the associated Lie system in G/H allows us to reduce the problem to one in the subgroup H.

Theorem: *Each solution of (•) on the group G can be written in the form $g(t) = g_1(t)\,h(t)$, where $g_1(t)$ is a curve on G projecting onto a solution $\tilde{g}_1(t)$ for the left action λ on the homogeneous space G/H and $h(t)$ is a solution of an equation but for the subgroup H, given explicitly by*

$$(\dot{h}\,h^{-1})(t) = -\mathrm{Ad}\,(g_1^{-1}(t)) \left(\sum_{\alpha=1}^{r} b_\alpha(t) a_\alpha + (\dot{g}_1\,g_1^{-1})(t) \right) \in T_e H \ .$$

The SODE Lie systems

A system of second order differential equations

$$\ddot{x}^i = f^i(t, x, \dot{x})\,, \qquad i = 1, \ldots, n,$$

can be studied through the corresponding system of first order differential equations

$$\begin{cases} \dfrac{dx^i}{dt} & = & v^i \\ \dfrac{dv^i}{dt} & = & f^i(t, x, v) \end{cases}$$

with associated t-dependent vector field

$$X = v^i \frac{\partial}{\partial x^i} + f^i(t, x, v) \frac{\partial}{\partial v^i}$$

We call SODE Lie systems those for which X is a Lie system, i.e. it can be written as a linear combination with t-dependent coefficients of vector fields closing a finite-dimensional real Lie algebra.

Examples

A) The 1-dim harmonic oscillator with time-dependent frequency

The equation of motion is

$$\ddot{x} = -\omega^2(t)x$$

with associated system

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -\omega^2(t)x \end{cases}$$

and vector field

$$X = v\frac{\partial}{\partial x} - \omega^2(t)x\,\frac{\partial}{\partial v}\ ,$$

which is a linear combination $X = X_1 - \omega^2(t)X_2$ with

$$X_1 = v\frac{\partial}{\partial x}\,, \qquad X_2 = x\,\frac{\partial}{\partial v}$$

such that if

$$X_3 = \frac{1}{2}\left(v\frac{\partial}{\partial v} - x\frac{\partial}{\partial x}\right)\ .$$

then

$$[X_1, X_2] = 2\,X_3\,, \quad [X_1, X_3] = -X_1\,, \quad [X_2, X_3] = X_2\,,$$

a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This system has no first integrals.

B) **The 2-dim isotropic harmonic oscillator with time-dependent frequency**

The equation of motion is

$$\left\{\begin{array}{rcl} \ddot{x}_1 & = & -\omega^2(t)x_1 \\ \ddot{x}_2 & = & -\omega^2(t)x_2 \end{array}\right.$$

with associated system

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & v_1 \\ \dot{v}_1 & = & -\omega^2(t)x_1 \\ \dot{x}_2 & = & v_2 \\ \dot{v}_2 & = & -\omega^2(t)x_2 \end{array}\right.$$

and the t-dependent vector field

$$X = v_1\frac{\partial}{\partial x_1} - \omega^2(t)x_1\,\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial x_2} - \omega^2(t)x_2\,\frac{\partial}{\partial v_2}\ ,$$

is a linear combination $X = X_1 - \omega^2(t)X_2$ with

$$X_1 = v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}\,, \qquad X_2 = x_1\frac{\partial}{\partial v_1} + x_2\frac{\partial}{\partial v_2}\ ,$$

such that if

$$X_3 = \frac{1}{2}\left(v_1\frac{\partial}{\partial v_1} - x_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial v_2} - x_2\,\frac{\partial}{\partial x_2}\right)\ .$$

then

$$[X_1, X_2] = 2\, X_3\,, \quad [X_1, X_3] = -X_1\,, \quad [X_2, X_3] = X_2\,,$$

once again a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The system admits an invariant because, if F is given by $F(x_1, x_2, v_1, v_2)$, then $X_1 F = 0$ shows that there exists a function $\bar{F}(\xi, v_1, v_2)$ with $\xi = x_1 v_2 - x_2 v_1$, such that $F(x_1, x_2, v_1, v_2) = \bar{F}(\xi, v_1, v_2)$ while the second condition

$$x_1\, \frac{\partial \bar{F}}{\partial v_1} + x_2\, \frac{\partial \bar{F}}{\partial v_2} = 0$$

i.e. we obtain the first integral

$$F = x_1 v_2 - x_2 v_1$$

which can be seen as a partial superposition rule.

With three copies of the same harmonic oscillator, the vector fields X_1 and X_2 are

$$X_1 = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v \frac{\partial}{\partial x}\,, \qquad X_2 = x_1 \frac{\partial}{\partial v_1} + x_2 \frac{\partial}{\partial v_2} + x \frac{\partial}{\partial v}\,,$$

which determine the first integrals F as solutions of $X_1F = X_2F = 0$. The condition $X_1F = 0$ says that there exists a function $\bar{F} : \mathbb{R}^5 \to \mathbb{R}^2$ such that $F(x_1, x_2, x, v_1, v_2, v) = \bar{F}(\xi_1, \xi_2, v_1, v_2, v)$ with $\psi_1(x_1, x_2, x, v_1, v_2, v) = xv_1 - x_1v$ and $\psi_2(x_1, x_2, x, v_1, v_2, v) = xv_2 - x_2v$, and the condition $X_2F = 0$ transforms into

$$x_1 \frac{\partial \bar{F}}{\partial v_1} + x_2 \frac{\partial \bar{F}}{\partial v_2} + x \frac{\partial \bar{F}}{\partial v}$$

i.e. ξ_1 and ξ_2 are first integrals. They produce a superposition rule, because

$$\left\{ \begin{array}{lcl} xv_2 - x_2v & = & k_1 \\ x_1v - v_1x & = & k_2 \end{array} \right.$$

from where we obtain the expected superposition rule:

$$x = k_1\, x_1 + k_2\, x_2\,, \qquad v = C_1\, v_1 + C_2\, v_2\,, \qquad C_i = \frac{k_i}{x_1v_2 - x_2v_1}$$

C) **Pinney equation**:

The Pinney equation is the following second order non-linear differential equation:

$$\ddot{x} = -\omega^2(t)x + \frac{k}{x^3}\,,$$

where k is a constant. The corresponding system of first order differential eqs is

$$\begin{cases} \dot{x} &= v \\ \dot{v} &= -\omega^2(t)x + \dfrac{k}{x^3} \end{cases}$$

and the associated t-dependent vector field

$$X = v\frac{\partial}{\partial x} + \left(-\omega^2(t)x + \frac{k}{x^3}\right)\frac{\partial}{\partial v}\,.$$

This is a Lie system because it can be written as

$$X = L_2 - \omega^2(t)L_1\,,$$

where:

$$L_1 := x\frac{\partial}{\partial v}, \quad L_2 = \frac{k}{x^3}\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}.$$

The vector fields L_1 and L_2 span a three-dimensional real Lie algebra $\mathfrak{g}$ with nonzero defining relations:

$$[L_1, L_2] = 2L_3, \quad [L_3, L_2] = -L_2, \quad [L_3, L_1] = L_1$$

where

$$L_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}\right),$$

which is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

The fact that they have the same associated Lie algebra means that they can be solved simultaneously in the group $SL(2, \mathbb{R})$ by the equation

$$\dot{g}\, g^{-1} = \omega^2(t)\, a_1 - a_2$$

Note that this isotonic oscillator shares with the harmonic one the property of having a period independent of the energy, i.e. they are isochronous, and in the quantum case they have a equispaced spectrum.

D) **Ermakov system**

Consider the system

$$\begin{cases} \dot{x} &= v_x \\ \dot{v}_x &= -\omega^2(t)x \\ \dot{y} &= v_y \\ \dot{v}_y &= -\omega^2(t)y + \dfrac{1}{y^3} \end{cases}$$

with associated vector field

$$X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} - \omega^2(t)x \frac{\partial}{\partial v_x} + \left(-\omega^2(t)y + \frac{1}{y^3}\right) \frac{\partial}{\partial v_y},$$

which is a linear combination with time-dependent coefficients of the vector fields, $X = -\omega^2(t)X_1 + X_2$, of the vector fields

$$X_1 = x \frac{\partial}{\partial v_x} + y \frac{\partial}{\partial v_y}, \qquad X_2 = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + \frac{1}{y^3} \frac{\partial}{\partial v_y}.$$

This system is made up by two Lie systems closing on a $\mathfrak{sl}(2,\mathbb{R})$ algebra.

The second subsystem of first order differential equations is usually called Pinney equation. The generators of the Lie system with algebra $\mathfrak{sl}(2,\mathbb{R})$ span a distribution of dimension two and there is no first integral of the motion for such subsystem.

By adding the other $\mathfrak{sl}(2,\mathbb{R})$ linear Lie system, the h.o. with time dependent angular frequency, as the distribution in the four-dimensional space is of rank three there is an integral of motion.

The first integral can be obtained from $X_1F = X_2F = 0$. But X_1F means that $F(x,y,v_x,v_y) = \bar{F}(x,y,\xi)$ with $\xi = xv_y - yv_x$, and then $X_2F = 0$ is written

$$v_x \frac{\partial \bar{F}}{\partial x} + v_x \frac{\partial \bar{F}}{\partial x} + \frac{x}{y^3}\frac{\partial \bar{F}}{\partial \xi}$$

and the associated characteristics system we obtain

$$\frac{x\,dy - y\,dx}{\xi} = \frac{y^3\,d\xi}{x} \Longrightarrow \frac{d(x/y)}{\xi} + \frac{y\,d\xi}{x} = 0$$

from where and the following first integral is found:

$$\psi(x,y,v_x,v_y) = \left(\frac{x}{y}\right)^2 + \xi^2 = \left(\frac{x}{y}\right)^2 + (xv_y - yv_x)^2$$

which is the well-known Ermakov invariant.

E) **Generalized Ermakov system**

It is the system given by:

$$\begin{cases} \ddot{x} &=& \dfrac{1}{x^3} f(y/x) - \omega^2(t) x \\ \ddot{y} &=& \dfrac{1}{y^3} g(y/x) - \omega^2(t) y \end{cases}$$

In the particular case $f(u) = 0$ and $g(u) = 1$ reduces to the Ermakov system.

This system can be written as a first order one by doubling the number of degrees of freedom by introducing the new variables v_x and v_y:

$$\begin{cases} \dot{x} &=& v_x \\ \dot{v}_x &=& -\omega^2(t) x + \dfrac{1}{x^3} f(y/x) \\ \dot{y} &=& v_y \\ \dot{v}_y &=& -\omega^2(t) y + \dfrac{1}{y^3} g(y/x) \end{cases}$$

which determines the integral curves of the vector field

$$X = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial v_y} + \left(-\omega^2(t) x + \frac{1}{x^3} f(y/x)\right) \frac{\partial}{\partial v_x} + \left(-\omega^2(t) y + \frac{1}{y^3} g(y/x)\right) \frac{\partial}{\partial v_y} \, .$$

Such vector field can be written as a linear combination

$$X = N_2 - \omega^2(t)\, N_1$$

where N_1 and N_2 are the vector fields

$$N_1 = x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y}, \quad N_2 = v_x\frac{\partial}{\partial x} + \frac{1}{x^3}f(y/x)\frac{\partial}{\partial v_x} + v_y\frac{\partial}{\partial y} + \frac{1}{y^3}g(y/x)\frac{\partial}{\partial v_y},$$

Note that these vector fields generate a three-dimensional real Lie algebra with a third generator

$$N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y}\right).$$

In fact, as

$$[N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_3, N_2] = -N_2$$

they generate a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Therefore the system is a Lie system.

There exists a first integral for the motion, $F : \mathbb{R}^4 \to \mathbb{R}$, for any $\omega^2(t)$, because this Lie system has an associated integrable distribution of rank three and the manifold is four-dimensional.

This first integral F satifies $N_i F = 0$ for $i = 1, \ldots, 3$, but as $[N_1, N_2] = 2N_3$ it is enough to impose $N_1 F = N_2 F = 0$. Then, if $N_1 F = 0$,

$$x\,\frac{\partial F}{\partial v_x} + y\,\frac{\partial F}{\partial v_y} = 0\,,$$

and according to the method of characteristics we obtain:

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dv_x}{x} = \frac{dv_y}{y}$$

and therefore there exists a function $\bar{F} : \mathbb{R}^3 \to \mathbb{R}$ such that $F(x, y, v_x, v_y) = \bar{F}(x, y, \xi = xv_y - yv_x)$. The condition $N_2 F = 0$ reads now

$$v_x \frac{\partial \bar{F}}{\partial x} + v_y \frac{\partial \bar{F}}{\partial y} + \left(-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x) \right) \frac{\partial \bar{F}}{\partial \xi}\,.$$

We can therefore consider the associated system the characteristics are given by:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{d\xi}{-\frac{y}{x^3} f(y/x) + \frac{x}{y^3} g(y/x)}$$

But using that

$$\frac{-y\,dx + x\,dy}{\xi} = \frac{dx}{v_x} = \frac{dy}{v_y}$$

we arrive to

$$\frac{-y\,dx + x\,dy}{\xi} = \frac{d\xi}{-\frac{y}{x^3}f(\frac{y}{x}) + \frac{x}{y^3}g(\frac{y}{x})}$$

i.e.

$$-\frac{y^2 d\left(\frac{x}{y}\right)}{\xi} = \frac{d\xi}{-\frac{y}{x^3}f(\frac{y}{x}) + \frac{x}{y^3}g(\frac{y}{x})}$$

and integrating we obtain the first integral

$$\frac{1}{2}\xi^2 + \int^u \left[\frac{1}{u^3}\,f\left(\frac{1}{u}\right) + u\,g\left(\frac{1}{u}\right)\right]\,du\,.$$

This first integral allows us to determine a solution of one subsystem in terms of a solution of the other equation.

F) The Pinney equation revisited

Consider the system of first order differential equations:

$$\begin{cases} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \\ \dot{v}_x &= -\omega^2(t)x \\ \dot{v}_y &= -\omega^2(t)y + \dfrac{k}{y^3} \\ \dot{v}_z &= -\omega^2(t)z \end{cases}$$

which corresponds to the vector field

$$X = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} + \frac{k}{y^3}\frac{\partial}{\partial v_y} - \omega^2(t)\left(x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial v_z}\right)$$

X can be expressed as $X = N_2 - \omega^2(t)N_1$ where the vector fields N_1 and N_2 are:

$$N_1 = y\frac{\partial}{\partial v_y} + x\frac{\partial}{\partial v_x} + z\frac{\partial}{\partial v_z}, \quad N_2 = v_y\frac{\partial}{\partial y} + \frac{1}{y^3}\frac{\partial}{\partial v_y} + v_x\frac{\partial}{\partial x} + v_z\frac{\partial}{\partial z},$$

These vector fields generate a three-dimensional real Lie algebra with the vector field N_3 given by

$$N_3 = \frac{1}{2}\left(x\frac{\partial}{\partial x} - v_x\frac{\partial}{\partial v_x} + y\frac{\partial}{\partial y} - v_y\frac{\partial}{\partial v_y} + z\frac{\partial}{\partial z} - v_z\frac{\partial}{\partial v_z}\right).$$

In fact, as

$$[N_1, N_2] = 2N_3, \quad [N_3, N_1] = N_1, \quad [N_3, N_2] = -N_2$$

they generate a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The system is a Lie system.

The distribution generated by these fundamental vector fields has rank three. Thus, as the manifold of the Lie system is of dimension six we obtain three time-independent integrals of motion.

- The Ermakov invariant I_1 of the subsystem involving variables x and y.
- The Ermakov invariant I_2 of the subsystem involving variables y and z
- The Wronskian W of the subsystem involving variables x and z has

They define a foliation with three-dimensional leaves.

We can use this foliation for obtaining a superposition rule in terms of these three first integrals.

The Ermakov invariants read as:

$$I_1 = \frac{1}{2}\left((yv_x - xv_y)^2 + c\left(\frac{x}{y}\right)^2\right)$$
$$I_2 = \frac{1}{2}\left((yv_z - zv_y)^2 + c\left(\frac{z}{y}\right)^2\right)$$

and W is:

$$W = x_1 v_{v_z} - zv_x$$

In terms of these three integrals we can obtain an explicit expression of y in terms of x, z and the integrals I_1, I_2, W:

$$y = \frac{2}{W}\left(I_2 x^2 + I_1 z^2 \pm \sqrt{4I_1 I_2 - cw^2}xz\right)^{1/2}$$

This can be interpreted as saying that there is a superposition rule allowing us to express the general solution of the Pinney equation in terms of two independent solutions of the corresponding harmonic oscillator with time-dependent frequency

Structure preserving Lie systems

There are particularly interesting cases in which the manifold M is endowed with additional structures. For instance, let (M,Ω) be a symplectic manifold and the vector fields arising in the expression of the t-dependent vector field describing a Lie system are Hamiltonian vector fields closing on a real finite-dimensional Lie algebra.

These vector fields correspond to a symplectic action of the Lie group G on (M,Ω).

The Hamiltonian functions of such vector fields, defined by $i(X_\alpha)\Omega = -dh_\alpha$, do not close on the same Lie algebra when the Poisson bracket is considered, but we can only say that

$$d\left(\{h_\alpha, h_\beta\} - h_{[X_\alpha, X_\beta]}\right) = 0\ ,$$

and then they span a Lie algebra extension of the original one.

The important fact is that we can define a t-dependent Hamiltonian

$$h_t = \sum_\alpha b_\alpha(t)\, h_\alpha,$$

with the functions h_α closing a Lie algebra, in such a wat hat $i(X_t)\Omega = -dh_t$.

As an example we can consider the differential equation of an n-dimensional Winternitz–Smorodinsky oscillator of the form

$$\left\{\begin{array}{rcl} \dot{x}_i &=& p_i, \\ \dot{p}_i &=& -\omega^2(t)x_i + \dfrac{k}{x_i^3}, \end{array}\right. \qquad i = 1, \ldots, n.$$

which describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}^n$

$$X_t = \sum_{i=1}^n \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t)x_i + \frac{k}{x_i^3}\right)\frac{\partial}{\partial p_i}\right],$$

which can be written as $X_t = X_2 + \omega^2(t)X_1$ with X_1, X_2 and $X_3 = -[X_1, X_2]$ being given by

$$X_1 = -\sum_{i=1}^n x_i \frac{\partial}{\partial p_i}, \quad X_2 = \sum_{i=1}^n \left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3}\frac{\partial}{\partial p_i}\right), \quad X_3 = \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} - p_i \frac{\partial}{\partial p_i}\right).$$

Note that X_t is a Lie system, because X_1, X_2 and X_3 close on a $\mathfrak{sl}(2,\mathbb{R})$ algebra:

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = X_1, \qquad [X_2, X_3] = -X_2.$$

Moreover, the preceding vector fields are Hamiltonian vector fields with respect to the usual symplectic form $\omega_0 = \sum_{i=1}^{n} dx^i \wedge dp_i$ with Hamiltonian functions

$$h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad h_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \frac{k}{x_i^2}\right), \qquad h_3 = \sum_{i=1}^{n} x_i p_i,$$

which obey that

$$\{h_1, h_2\} = h_3, \qquad \{h_1, h_3\} = -h_1, \qquad \{h_2, h_3\} = h_2.$$

Consequently, every curve h_t that takes values in the Lie algebra $(W, \{\cdot, \cdot\})$ spanned by h_1, h_2 and h_3 gives rise to a Lie system which is Hamiltonian in $\mathrm{T}^*\mathbb{R}^n$ with respect to the symplectic structure ω_0 in such a way that the t-dependent vector field is given by

$$X_t = X_2 + \omega^2(t) X_1 = \widehat{\omega}_0^{-1}(dh_2 + \omega^2(t) dh_1),$$

i.e. the Hamiltonian is $h_t = h_2 + \omega^2(t) h_1$.

We can go a step further and consider Lie systems in (may be degenerate) Poisson manifolds.

Definition. *A Poisson manifold is a pair (M, Λ) where Λ is a bivector field in the differentiable manifold M in such a way that $[\Lambda, \Lambda]_{\text{S.B.}} = 0$. The bivector field gives by contraction a map denoted $\widehat{\Lambda}$ such that*

$$\widehat{\Lambda}(\alpha)(\beta) = \Lambda(\alpha, \beta)$$

In particular, if $f_1, f_2 \in C^\infty(M)$, we define the Poisson bracket $\{f_1, f_2\}$ by

$$\{f_1, f_2\} = \Lambda(df_1, df_2),$$

and this Poisson bracket satisfies Jacobi identity because of the vanishing of the Schouten bracket condition

The Lie bracket over $C^\infty(M)$ holds the Leibnitz rule

$$\{fg, h\} = \{f, h\}g + \{g, h\}f, \qquad \forall f, g, h \in C^\infty(M).$$

Consequently, the above Lie bracket becomes a derivation in each entry: given a function $f \in C^\infty(M)$, there exists a vector field X_f over M such that $X_f g = \{g, f\}$ for each $g \in C^\infty(M)$, i.e. $X_f = \widehat{\Lambda}(-df)$. T he vector field X_f is called the Hamiltonian vector field associated with f. The Jacobi identity for the Poisson structure entails that

$$X_{\{f,g\}} = -[X_f, X_g], \qquad \forall f, g \in C^\infty(M).$$

In other words, the mapping $f \mapsto X_f$ is a Lie algebra anti-homomorphism between the Lie algebras $(C^\infty(M), \{\cdot,\cdot\})$ and $(\Gamma(\tau_M), [\cdot,\cdot])$.

Equivalently, $\widehat{\Lambda} \circ d : C^\infty(M) \to \mathfrak{X}_H(M, \Lambda)$ is a Lie algebra homomorphism.

Definition. *The elements of the kernel of the previous homomorphism are called Casimir functions. The set of such Casimir functions will be denoted $\mathcal{C}$.*

This can be summarising by saying that the following sequence is exact:

$$0 \longrightarrow \mathcal{C} \longrightarrow C^\infty(M) \xrightarrow{\widehat{\Lambda}\circ d} \mathfrak{X}_H(M, \Lambda) \longrightarrow 0$$

Definition. *A Lie–Hamiltonian structure is a triple (M, Λ, h), where (M, Λ) is a Poisson manifold and h is a t-parametrised family of functions $h_t : M \to \mathbb{R}$ such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ is a finite-dimensional real Lie algebra.*

Definition. *A t-dependent system X on M is said to admit a Lie–Hamilton structure if there exists a Lie–Hamiltonian structure (M, Λ, h) such that $X_t \in \widehat{\Lambda}(-dh_t)$, for every $t \in \mathbb{R}$. The triple (M, Λ, X) is called a Lie–Hamilton triple.*

There is a generalisation to the framework of Dirac manifolds. Recall that a Pontryagin bundle $\mathcal{P}N$ is a vector bundle $TN \oplus_N T^*N$ on N, and that an almost-Dirac manifold is a pair (N, L), where L is a maximally isotropic subbundle of $\mathcal{P}N$ with respect to the pairing

$$\langle X_x + \alpha_x, \bar{X}_x + \bar{\alpha}_x \rangle_+ \equiv \frac{1}{2}(\bar{\alpha}_x(X_x) + \alpha_x(\bar{X}_x)),$$

where $X_x + \alpha_x, \bar{X}_x + \bar{\alpha}_x \in T_xN \oplus T^*_xN = \mathcal{P}_xN$, i.e., L is isotropic and has rank $n = \dim N$.

A Dirac manifold is an almost-Dirac manifold (N, L) whose subbundle L, its Dirac structure, is involutive relative to the Courant–Dorfman bracket, namely

$$[[X + \alpha, \bar{X} + \bar{\alpha}]]_C \equiv [X, \bar{X}] + \mathcal{L}_X \bar{\alpha} - \iota_{\bar{X}} d\alpha \,,$$

where $X + \alpha, \bar{X} + \bar{\alpha} \in \Gamma(TN \oplus_N T^*N)$.

A vector field X on N is said to be an L-Hamiltonian vector field (or simply a Hamiltonian vector field if L is fixed) if there exists an $f \in C^\infty(N)$ such that $X + df \in \Gamma(L)$. In this case, f is an L-Hamiltonian function for X and an admissible function of (N, L). Let us denote by $\mathrm{Ham}(N, L)$ and $\mathrm{Adm}(N, L)$ the spaces of Hamiltonian vector fields and admissible functions of (N, L), respectively.

The space $\mathrm{Adm}(N,L)$ becomes a Poisson algebra $(\mathrm{Adm}(N,L), \bullet, \{\cdot,\cdot\}_L)$ relative to the standard product of functions and the Lie bracket given by

$$\{f,\bar{f}\}_L = X\bar{f}\,,$$

where X is an L-Hamiltonian vector field for f.

Moreover, if X and $\bar{X}$ are L-Hamiltonian vector fields with Hamiltonian functions f and $\bar{f}$, then $\{f,\bar{f}\}_L$ is a Hamiltonian for $[X,\bar{X}]$:

$$[[X+df,\bar{X}+d\bar{f}]]_C = [X,\bar{X}] + \mathcal{L}_X d\bar{f} - \iota_{\bar{X}} d^2 f = [X,\bar{X}] + d\{f,\bar{f}\}_L.$$

One can proceed in a very a similar way to the case of Poisson manifolds

See e.g.

Dirac–Lie systems and Schwarzian equations, J. Diff. Eqns. **257**, 2303–2340 (2014) (JFC, Janusz Grabowski, Javier de Lucas and Cristina Sardón)

An example: Second order Riccati differential equation

The usual Riccati equation comes from reduction of a linear differential equation by taking into account the invariance under dilations of such equations.

Starting from

$$A_3\, \dddot{y} + A_2\, \ddot{y} + A_1\, \dot{y} + A_0\, y = 0$$

where we can assume that $A_3(t) > 0$, and writing $y = e^u$, with $x = \dot{u}$ we arrive to

$$A_3(\ddot{x} + 3\, x\dot{x} + x^3) + A_2(\dot{x} + x^2) + A_1\, x + A_0 = 0,$$

and if we change the independent variable t to a new variable τ, then $d/dt = \dot{\tau}\, d/d\tau$, and if we denote $x' = dx/d\tau$, $x'' = d^2x/d\tau^2$, we obtain

$$\dot{x} = \dot{\tau}\, x', \quad \ddot{x} = \dot{\tau}\frac{d}{d\tau}\left(\dot{\tau}\,\frac{dx}{d\tau}\right) = \dot{\tau}^2\, x'' + \frac{\ddot{\tau}}{\dot{\tau}}\, x'$$

and therefore the original equation reduces to

$$A_3\left(\dot{\tau}^2\, x'' + \frac{\ddot{\tau}}{\dot{\tau}}\, x' + 3\, x\dot{\tau}\, x' + x^3\right) + A_2\left(\dot{\tau}x' + x^2\right) + A_1\, x + A_0 = 0.$$

If we choose τ such that $A_3\,\dot\tau^2=1$, and therefore

$$\dot\tau=A_3^{-1/2}\Longrightarrow\ddot\tau=-\frac{1}{2}A_3^{-3/2}\dot A_3,\qquad \frac{\ddot\tau}{\dot\tau}=-\frac{1}{2}A_3^{-1}\,\dot A_3,$$

we find the equation

$$x''-\frac{1}{2}A_3^{-1}\dot A_3\,x'+3A_3^{-1/2}x\,x'+A_3\,x^3+A_2A_3^{-1}\,x'+A_2\,x^2+A_1\,x+A_0=0,$$

which can be rewritten in he form:

$$\ddot x+(b_0(t)+b_1(t)x)\dot x+c_0(t)+c_1(t)x+c_2(t)x^2+c_3(t)x^3=0,$$

with

$$b_1(t)=3\sqrt{A_3(t)},\qquad b_0(t)=\frac{A_2(t)}{\sqrt{A_3(t)}}-\frac{\dot A_3(t)}{2A_3(t)},$$

and is considered as the most general second orden Riccati equation.

It has been shown (JFC+ MF Rañada+M Santander, JMP **46**, 062703 (2005)) that such second-order Riccati equations admit a Lagrangian of the form:

$$L(t,x,v)=\frac{1}{v+U(t,x)},$$

with $U(t,x)=a_0(t)+a_1(t)x+a_2(t)x^2$.

The corresponding t-dependent Hamiltonian obtained from the Legendre transformation

$$p=\frac{\partial L}{\partial v}=-\frac{1}{(v+U(t,x))^2}\Longrightarrow v=\frac{1}{\sqrt{-p}}-U(t,x),$$

i.e. the image is the open submanifold $\mathcal{O}=\{(x,p)\in \mathrm{T}^*_x\mathbb{R}\mid p<0\}$ and we can define in $\mathcal{O}$ the Hamiltonian

$$h(t,x,p)=p\left(\frac{1}{\sqrt{-p}}-U(t,x)\right)-\sqrt{-p}=-2\sqrt{-p}-p\,U(t,x).$$

Consequently, the Hamilton equations for h are

$$\begin{cases}\dot{x} &= \dfrac{\partial h}{\partial p}=\dfrac{1}{\sqrt{-p}}-U(t,x),\\ \dot{p} &= -\dfrac{\partial h}{\partial x}=p\dfrac{\partial U}{\partial x}(t,x).\end{cases}$$

which, taking into account the form of $U(t,x)$ turn out to be

$$\begin{cases}\dot{x} &= \dfrac{\partial h}{\partial p}=\dfrac{1}{\sqrt{-p}}-a_0(t)-a_1(t)x-a_2(t)x^2,\\ \dot{p} &= -\dfrac{\partial h}{\partial x}=p(a_1(t)+2a_2(t)x).\end{cases}$$

This is a Lie system: In fact, consider the set of vector fields

$$X_1 = \frac{1}{\sqrt{-p}}\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p},$$
$$X_4 = x^2\frac{\partial}{\partial x} - 2xp\frac{\partial}{\partial p}, \quad X_5 = \frac{x}{\sqrt{-p}}\frac{\partial}{\partial x} + 2\sqrt{-p}\frac{\partial}{\partial p}.$$

The time-dependent vector field describing the system is

$$X(t,x) = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4,$$

and the vector fields close on the commutation relations

$$\begin{array}{llll} [X_1, X_2] = 0, & [X_1, X_3] = \frac{1}{2}X_1, & [X_1, X_4] = X_5, & [X_1, X_5] = 0, \\ [X_2, X_3] = X_2, & [X_2, X_4] = 2X_3, & [X_2, X_5] = X_1, & \\ [X_3, X_4] = X_4, & [X_3, X_5] = \frac{1}{2}X_5, & & \\ [X_4, X_5] = 0. & & & \end{array}$$

and then we see that it is a Lie system related to a Vessiot-Guldberg Lie algebra of vector fields V.

More specifically, the vector fields $X_1, \ldots, X_5$ span a five dimensional Lie algebra of vector fields V that is not solvable because $[V, V] = V$.

Moreover, V is not a semisimple algebra. It admits an Abelian solvable ideal $V_1 = \langle X_1, X_5 \rangle)$, and $V_2 = \langle X_2, X_3, X_4 \rangle$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Therefore V is a semidirect sum $V_1 \oplus_s V_2$.

Consequently, the Lie algebra V gives rise to a Lie group of the form $G = \mathbb{R}^2 \lhd SL(2, \mathbb{R})$, where $\lhd$ denotes the semidirect product of $SL(2, \mathbb{R})$ by $\mathbb{R}^2$, and a Lie group action $\Phi : G \times \mathcal{O} \to \mathcal{O}$ whose fundamental vector fields are those of V.

Indeed, it is a long, but straightforward computation, to show that

$$\Phi\left(\left((\lambda_1, \lambda_2), \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right), (x, p)\right) = \left(\frac{\bar{x} - \lambda_1 \sqrt{-\bar{p}}}{1 + \lambda_5 (-\bar{p})^{-1/2}}, -(\sqrt{-\bar{p}} + \lambda_5)^2\right),$$

where

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \bar{p} = p\,(\delta + \gamma x)^2\,.$$

This action enables us to put the general solution $\xi(t)$ of the system of Hamilton equations for the second order Riccati equation in the form $\xi(t)=\Phi(g(t),\xi_0)$, where $g(t)$ is the solution of the equation

$$\frac{dg}{dt}=-\sum_{\alpha=1}^{5} b_\alpha(t)X_\alpha^R(g), \qquad g(0)=e,$$

on G, with the X_α^R being a family of right-invariant vector fields over G such that the $X_\alpha^R(e)\in T_eG$ close the same commutation relations as the X_α.

To be remarked that the vector fields X_i here considered are Hamiltonian with respect to the usual symplectic form in $\mathrm{T}^*\mathbb{R}$, their hamiltonians being respectively given by:

$$h_1=2\sqrt{-p}, \quad h_2=-p, \quad h_3=-xp, \quad h_4=-x^2p,$$

and it turns out that their nonvanishing Poisson brackets are

$$\{h_1,h_3\}=\tfrac{1}{2}h_1, \quad \{h_1,h_4\}=h_5, \quad \{h_1,h_5\}=2, \quad \{h_2,h_3\}=h_2,$$

$$\{h_2,h_4\}=2h_3, \quad \{h_2,h_5\}=h_1, \quad \{h_3,h_4\}=h_4, \quad \{h_3,h_5\}=\tfrac{1}{2}h_5$$

with $h_5=2x\sqrt{-p}$. They close on a six-dimensional real Lie algebra with the function $h_6=1$. Moreover, it can be seen that the t-dependent system can be put into the form $\widehat{\Lambda}(-dh_t)$, where h_t is a t-parametrized family of functions over $\mathcal{O}$ of the form $h_t = h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2h_4$ and therefore the Lie system we are considering is Hamiltonian

Finally, a superposition rule for the second order Riccati equation can be obtained through the common first-integrals for the appropriated diagonal prolongations $\widehat{X}_1, \widehat{X}_2, \widehat{X}_3, \widehat{X}_4, \widehat{X}_5$ on a certain $\mathcal{O}^{(m)} \subset \mathrm{T}^*\mathbb{R}^{(m)}$ (i.e. such that their projections $\pi_*(\widehat{X}_\alpha)$, with $\alpha = 1, \ldots, 5$, are linearly independent at a generic point of $\mathrm{T}^*\mathbb{R}^m$). In our case, it can be easily verified that $m = 4$. The resulting first-integrals, turn out to be

$$\Delta_1 = (x_{(2)} - x_{(3)})\sqrt{p_{(2)}p_{(3)}} + (x_{(3)} - x_{(1)})\sqrt{p_{(3)}p_{(1)}} + (x_{(1)} - x_{(2)})\sqrt{p_{(2)}p_{(1)}},$$

$$\Delta_2 = (x_{(1)} - x_{(2)})\sqrt{p_{(1)}p_{(2)}} + (x_{(2)} - x_{(0)})\sqrt{p_{(2)}p_{(0)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(1)}p_{(0)}},$$

$$\Delta_3 = (x_{(1)} - x_{(3)})\sqrt{p_{(1)}p_{(3)}} + (x_{(3)} - x_{(0)})\sqrt{p_{(3)}p_{(0)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(1)}p_{(0)}}.$$

In order to obtain a superposition principle, we just need to obtain the value of $p_{(0)}$ in terms of the remaining variables from one of the above integrals, e.g. Δ_2, to get

$$p_{(0)} = \frac{\Delta_2 + (x_{(2)} - x_{(1)})\sqrt{p_{(1)}p_{(2)}}}{(x_{(2)} - x_{(1)})\sqrt{p_{(2)}} + (x_{(0)} - x_{(1)})\sqrt{p_{(1)}}},$$

and to plug this value in one of the others variables, e.g. Δ_3, to have

$$\begin{aligned} x_{(0)} &= \frac{\Delta_3(\sqrt{-p_2}x_2-\sqrt{-p_1}x_1)+\Delta_2(\sqrt{-p_1}x_1-\sqrt{-p_3}x_3)+\Delta_1 x_1\sqrt{-p_1}}{\Delta_3(\sqrt{-p_2}-\sqrt{-p_1})+\Delta_2(\sqrt{-p_1}-\sqrt{-p_3})+\sqrt{-p_1}k_1}, \\ p_{(0)} &= \frac{(\Delta_1\sqrt{-p_1}+\Delta_3(\sqrt{-p_1}-\sqrt{-p_2})+\Delta_2(\sqrt{-p_1}-\sqrt{-p_3}))(\Delta_2+\sqrt{p_1p_2}(x_2-x_1))}{\Delta_1\sqrt{p_2p_1}(x_1-x_2)+\Delta_2\sqrt{p_2p_3}(x_2-x_3)+\sqrt{p_3p_2}(x_3-x_2)+\Delta_1}. \end{aligned}$$

The above expression gives us a superposition rule for second order Riccati differential equation.

In addition, as its general solution, $(x_{(0)}(t), p_{(0)}(t))$, satisfies that $x_{(0)}(t)$ is the general solution, the first part of the above expressions gives us the solution of second-order Riccati equations in terms of three particular solutions $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$, their associated moments $p_{(1)}(t), p_{(2)}(t), p_{(3)}(t)$, and two constants Δ_1, Δ_2

Note that once a family of particular solutions is chosen the constant Δ_1 gets fixed.

Geometric approach to Quantum Mechanics

The Schrödinger picture of Quantum mechanics admits a geometric interpretation similar to that of classical mechanics.

A separable complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ can be considered as a real linear space, to be then denoted $\mathcal{H}_{\mathbb{R}}$. The norm in $\mathcal{H}$ defines a norm in $\mathcal{H}_{\mathbb{R}}$, where $\|\psi\|_{\mathbb{R}} = \|\psi\|_{\mathbb{C}}$.

The linear real space $\mathcal{H}_{\mathbb{R}}$ is endowed with a natural symplectic structure as follows:

$$\omega(\psi_1, \psi_2) = 2\,\mathrm{Imag}\,\langle\psi_1, \psi_2\rangle.$$

The Hilbert $\mathcal{H}_{\mathbb{R}}$ can be considered as a real manifold modelled by a Banach space admitting a global chart.

The tangent space $T_\phi\mathcal{H}_{\mathbb{R}}$ at any point $\phi \in \mathcal{H}_{\mathbb{R}}$ can be identified with $\mathcal{H}_{\mathbb{R}}$ itself: the isomorphism associates $\psi \in \mathcal{H}_{\mathbb{R}}$ with the vector $\dot\psi \in T_\phi\mathcal{H}_{\mathbb{R}}$ given by:

$$\dot\psi f(\phi) := \left(\frac{d}{dt} f(\phi + t\psi)\right)_{|t=0}, \qquad \forall f \in C^\infty(\mathcal{H}_{\mathbb{R}})\ .$$

The real manifold can be endowed with a symplectic 2-form ω:

$$\omega_\phi(\dot{\psi}, \dot{\psi}') = 2\,\mathrm{Imag}\,\langle \psi, \psi' \rangle \ .$$

One can see that the constant symplectic structure ω in $\mathcal{H}_\mathbb{R}$, considered as a Banach manifold, is exact, i.e., there exists a 1-form $\theta \in \bigwedge^1(\mathcal{H}_\mathbb{R})$ such that $\omega = -d\theta$. Such a 1-form $\theta \in \bigwedge^1(\mathcal{H})$ is, for instance, the one defined by

$$\theta(\psi_1)[\dot{\psi}_2] = -\mathrm{Imag}\,\langle \psi_1, \psi_2 \rangle.$$

This shows that the geometric framework for usual Schrödinger picture is that of symplectic mechanics, as in the classical case.

A continuous vector field in $\mathcal{H}_\mathbb{R}$ is a continuous map $X\colon \mathcal{H}_\mathbb{R} \to \mathcal{H}_\mathbb{R}$. For instance for each $\phi \in \mathcal{H}$, the constant vector field X_ϕ defined by

$$X_\phi(\psi) = \dot{\phi}.$$

It is the generator of the one-parameter subgroup of transformations of $\mathcal{H}_\mathbb{R}$ given by

$$\Phi(t, \psi) = \psi + t\,\phi\,.$$

As another particular example of vector field consider the vector field X_A defined by the $\mathbb{C}$-linear map $A : \mathcal{H} \to \mathcal{H}$, and in particular when A is skew-selfadjoint.

With the natural identification natural of $T\mathcal{H}_\mathbb{R} \approx \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R}$, X_A is given by

$$X_A : \phi \mapsto (\phi, A\phi) \in \mathcal{H}_\mathbb{R} \times \mathcal{H}_\mathbb{R} \,.$$

When $A = I$ the vector field X_I is the Liouville generator of dilations along the fibres, $\Delta = X_I$, usually denoted Δ given by $\Delta(\phi) = (\phi, \phi)$.

Given a selfadjoint operator A in $\mathcal{H}$ we can define a real function in $\mathcal{H}_\mathbb{R}$ by

$$a(\phi) = \langle \phi, A\phi \rangle \,,$$

i.e.,

$$a = \langle \Delta, X_A \rangle \,.$$

Then,

$$\begin{aligned} da_\phi(\psi) &= \frac{d}{dt} a(\phi + t\psi)_{t=0} = \frac{d}{dt} \left[\langle \phi + t\psi, A(\phi + t\psi) \rangle \right]_{|t=0} \\ &= 2\,\mathrm{Re}\,\langle \psi, A\phi \rangle = 2\,\mathrm{Imag}\,\langle -\mathrm{i}\,A\phi, \psi \rangle = \omega(-\mathrm{i}\,A\phi, \psi). \end{aligned}$$

If we recall that the Hamiltonian vector field defined by the function a is such that for each $\psi \in T_\phi\mathcal{H} = \mathcal{H}$,

$$da_\phi(\psi) = \omega(X_a(\phi), \psi) \,,$$

we see that

$$X_a(\phi) = -\mathrm{i}\, A\phi\,.$$

Therefore if A is the Hamiltonian H of a quantum system, the Schrödinger equation describing time-evolution plays the rôle of 'Hamilton equations' for the Hamiltonian dynamical system $(\mathcal{H}, \omega, h)$, where $h(\phi) = \langle \phi, H\phi\rangle$: the integral curves of X_h satisfy

$$\dot{\phi} = X_h(\phi) = -\mathrm{i}\, H\phi\,.$$

The real functions $a(\phi) = \langle \phi, A\phi\rangle$ and $b(\phi) = \langle \phi, B\phi\rangle$ corresponding to two selfadjoint operators A and B satisfy

$$\{a, b\}(\phi) = -\mathrm{i}\,\langle \phi, [A, B]\phi\rangle\,,$$

because

$$\{a, b\}(\phi) = [\omega(X_a, X_b)](\phi) = \omega_\phi(X_a(\phi), X_b(\phi)) = 2\,\mathrm{Imag}\,\langle A\phi, B\phi\rangle\,,$$

and taking into account that

$$2\,\mathrm{Imag}\,\langle A\phi, B\phi\rangle = -\mathrm{i}\,[\langle A\phi, B\phi\rangle - \langle B\phi, A\phi\rangle] = -\mathrm{i}\,[\langle \phi, AB\phi\rangle - \langle \phi, BA\phi\rangle]\,,$$

we find the above result.

In particular, on the integral curves of the vector field X_h defined by a Hamiltonian H,

$$\dot{a}(\phi) = \{a, h\}(\phi) = -\mathrm{i}\,\langle \phi, [A, H]\phi \rangle\,,$$

what is usually known as Ehrenfest theorem:

$$\frac{d}{dt}\langle \phi, A\phi \rangle = -\mathrm{i}\,\langle \phi, [A, H]\phi \rangle\,.$$

There is another relevant symmetric $(0, 2)$ tensor field which is given by the Real part of the inner product. It endows $\mathcal{H}_{\mathbb{R}}$ with a Riemann structure and we have also a complex structure J such that

$$g(v_1, v_2) = -\omega(Jv_1, v_2), \qquad \omega(v_1, v_2) = g(Jv_1, v_2),$$

together with

$$g(Jv_1, Jv_2) = g(v_1, v_2), \qquad \omega(Jv_1, Jv_2) = \omega(v_1, v_2)\,.$$

The triplet (g, J, ω) defines a Kähler structure in $\mathcal{H}_{\mathbb{R}}$ and the symmetry group of the theory must be the unitary group $U(\mathcal{H})$ whose elements preserve the inner product, or in an alternative but equivalent way (in the finite-dimensional case), by the intersection of the orthogonal group $O(2n, \mathbb{R})$ and the symplectic group $Sp(2n, \mathbb{R})$.

The time evolution from time t_0 to time t, even in the non-autonomous case, is described in terms of the evolution operator $U(t,t_0)$:

$$\psi(t) = U(t,t_0)\psi(t_0)$$

It must be a symmetry of the theory, i.e. for each fixed t_0, $U(t,t_0)$ is a curve in the unitary group $U(\mathcal{H})$.

Assume by simplicity that $\mathcal{H}$ is finite-dimensional, and then as

$$\frac{dU(t,t_0)}{dt} \in T_{U(t,t_0)}U(\mathcal{H}) \Longrightarrow \frac{dU(t,t_0)}{dt}(U(t,t_0))^{-1} \in T_I U(\mathcal{H}) \approx \mathfrak{u}(\mathcal{H}),$$

and therefore, there exists a curve $H(t)$ in $\mathrm{Herm}(n,\mathbb{C})$ such that

$$\frac{dU(t,t_0)}{dt} = -\mathrm{i}\, H(t)\, U(t,t_0).$$

In this equation $H(t)$ does not depend on t_0 because of the relation

$$U(t,t_0) = U(t,t_1)U(t_1,t_0),$$

which implies

$$\frac{dU(t,t_0)}{dt}(U(t,t_0))^{-1} = \frac{dU(t,t_1)}{dt}(U(t,t_1))^{-1}$$

This is a Lie system in the unitary group $U(\mathcal{H})$ with associated Lie algebra $\mathfrak{u}(\mathcal{H})$ in the most general case. Sometimes however we can deal with some of its subalgebras.

Every curve $H(t)$ in $\mathfrak{u}(\mathcal{H})$ can be written as a linear combination of at most n^2 elements, those of a basis of $\mathfrak{u}(\mathcal{H})$, and therefore these (finite-dimensional) quantum systems are Lie systems.

As the elements of the Vessiot-Guldberg Lie algebra are skew-Hermitians, all of them define simultaneously Hamiltonian vector fields and Killing vector fields, and the system is a Lie-Kähler system.

As an example consider a Hamiltonian operator $H(t)$ that can be written as a linear combination, with some t-dependent real coefficients $b_1(t), \ldots, b_r(t)$, of some Hermitian operators,

$$H(t) = \sum_{k=1}^{r} b_k(t) H_k \,,$$

where the H_k form a basis of a real finite-dimensional Lie algebra V relative to the Lie bracket of observables, i.e. $[H_j, H_k] = \sum_{l=1}^{r} \mathrm{i}\, c_{jkl} H_l$, with $c_{jkl} \in \mathbb{R}$ and $j, k, l = 1, \ldots, r$.

It determines a t-dependent Schrödinger equation

$$\frac{d\psi}{dt} = -\mathrm{i}H(t)\psi = -\mathrm{i}\sum_{k=1}^{r} b_k(t)H_k\psi.$$

The vector fields X_k such that $X_k(\psi) = -\mathrm{i}\,H_k\,\psi$ are such that the t-dependent vector vector field X corresponding to the equation is $X = \sum_{k=1}^{r} b_k(t)X_k$ and

$$[X_j, X_k] = -\sum_{l=1}^{r} c_{jkl}X_l, \qquad j,k = 1,\ldots,r.$$

As an instance, if $\mathcal{H} = \mathbb{C}^2$, the time evolution is described by a curve $-\mathrm{i}H(t) := \dot{U}_t U_t^{-1}$ in the Lie algebra $\mathfrak{u}(2)$ of $U(2)$. Using the basis

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and denoting $\mathbf{S} = (\sigma_1, \sigma_2, \sigma_3)/2$ and $\mathbf{B} := (B_1, B_2, B_3)$, the Hamiltonian can be written as

$$H(t) := B_0(t)I_0 + \mathbf{B}(t)\cdot\mathbf{S}.$$

Using the identification of $\mathbb{C}^2$ with $\mathbb{R}^4$, the Schrödinger equation is

$$\begin{pmatrix} \dot q_1 \\ \dot p_1 \\ \dot q_2 \\ \dot p_2 \end{pmatrix} = \frac{1}{2}\left(\begin{matrix} 0 & 2B_0(t)+B_3(t) & -B_2(t) & B_1(t) \\ -2B_0(t)-B_3(t) & 0 & -B_1(t) & -B_2(t) \\ B_2(t) & B_1(t) & 0 & 2B_0(t)-B_3(t) \\ -B_1(t) & B_2(t) & B_3(t)-2B_0(t) & 0 \end{matrix}\right]\begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}.$$

while the vector fields are now

$$\begin{aligned} X_0 &= -\Gamma = p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}, \\ X_1 &= \frac{1}{2}\left(p_2\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_2} - q_1\frac{\partial}{\partial p_2}\right), \\ X_2 &= \frac{1}{2}\left(-q_2\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial p_1} + q_1\frac{\partial}{\partial q_2} + p_1\frac{\partial}{\partial p_2}\right), \\ X_3 &= \frac{1}{2}\left(p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1} - p_2\frac{\partial}{\partial q_2} + q_2\frac{\partial}{\partial p_2}\right) \end{aligned}$$

satisfying

$$[X_0, \cdot] = 0, \qquad [X_1, X_2] = -X_3, \qquad [X_2, X_3] = -X_1, \qquad [X_3, X_1] = -X_2.$$

The vector fields X_0, X_1, X_2, X_3 are Hamiltonian with Hamiltonian functions given by

$$\begin{aligned} h_0(\psi) &= \tfrac{1}{2}\langle\psi,\psi\rangle = \tfrac{1}{2}(q_1^2+p_1^2+q_2^2+p_2^2), \\ h_1(\psi) &= \tfrac{1}{2}\langle\psi,S_1\psi\rangle = \tfrac{1}{2}(q_1q_2+p_1p_2), \\ h_2(\psi) &= \tfrac{1}{2}\langle\psi,S_2\psi\rangle = \tfrac{1}{2}(q_1p_2-p_1q_2), \\ h_3(\psi) &= \tfrac{1}{2}\langle\psi,S_3\psi\rangle = \tfrac{1}{4}(q_1^2+p_1^2-q_2^2-p_2^2). \end{aligned}$$

h_1, h_2, h_3 are functionally independent, but $h_0^2 = 4(h_1^2+h_2^2+h_3^2)$.

When $\mathcal{H}$ is not finite-dimensional Lie system theory applies when the t-dependent Hamiltonian can be written as a linear combination with t-dependent coefficients of Hamiltonians H_i closing on, under the commutator bracket, a real finite-dimensional Lie algebra.

Note however that this Lie algebra does not necessarily coincide with the corresponding classical one, but it is a Lie algebra extension.

On the other hand, as the fundamental concept for measurements is the expectation value of observables, two vector fields such that

$$\frac{\langle\psi_2, A\psi_2\rangle}{\langle\psi_2, \psi_2\rangle} = \frac{\langle\psi_1, A\psi_1\rangle}{\langle\psi_1, \psi_1\rangle}, \quad \forall A \in \mathrm{Her}(\mathcal{H})$$

should be considered as indistinguishable.

This is only possible when ψ_2 is proportional to ψ_1, and therefore we must consider rays rather than vectors the elements describing the quantum states.

The space of states is not $\mathbb{C}^n$ but the projective space $\mathbb{CP}^{n-1}$.

It is possible to define a Kähler structure on $\mathbb{CP}^{n-1}$ and therefore to study Lie-Kähler systems leading to superposition rules and to study time evolution in this projective space.

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THANKS FOR YOUR ATTENTION !!!