Orbital measures and spline functions

Jacques Faraut

Bedlewo, 26 September 2016

For a Hermitian matrix $X \in Herm(n, \mathbb{C})$, the classical spectral theorem says that the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are real and the corresponding eigenspaces orthogonal.

The unitary group $U(n)$ acts on $Herm(n, \mathbb{C})$ by the transformations $X \mapsto uXu^*$ $(u \in U(n))$.
Let $\mathcal{O}_A$ denote the orbit of the matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_1 \leq \cdots \leq a_n$:

$$\mathcal{O}_A = \{X = uAu^* \mid u \in U(n)\}.$$

By the spectral theorem

$$\mathcal{O}_A = \big\{X \in Herm(n, \mathbb{C}) \mid \mathrm{spectrum}(X) = \{a_1, \ldots, a_n\}\big\}.$$

Let p be the projection of $Herm(n,\mathbb{C})$ onto $Herm(n-1,\mathbb{C})$ which maps the matrix X to the $(n-1)\times(n-1)$ upper left corner Y of X.

$$X=(x_{ij}),\quad Y=\begin{pmatrix} x_{11} & \dots & x_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ x_{n-1,1} & \dots & x_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Theorem of Rayleigh

The eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of the matrix $Y=p(X)$ interlace the sequence of the eigenvalues of X:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

This relation will be written $\mu \preceq \lambda$.

Sketch of the proof

One computes the rational function

$$f(z) = \big((zI_n - X)^{-1} e_n \mid e_n\big)$$

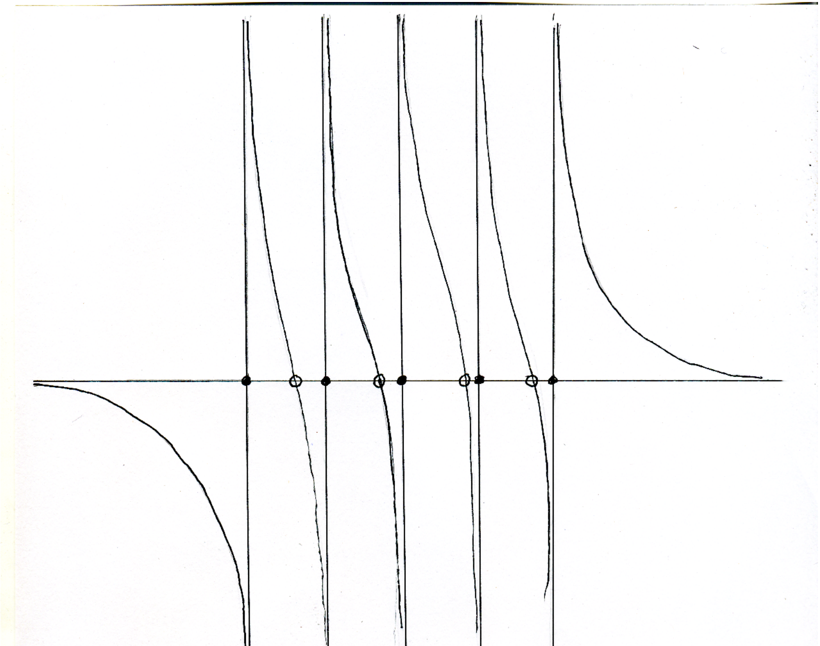
in two different ways

$$\begin{aligned} f(z) \qquad &= \frac{\det_{n-1}(zI_{n-1} - Y)}{\det_n(zI_n - X)} = \frac{\prod_{j=1}^{n-1}(z-\mu_j)}{\prod_{i=1}^{n}(z-\lambda_i)} \\ &= \sum_{i=1}^{n} \frac{w_i}{z-\lambda_i}. \end{aligned}$$

The poles are the λ_i, the zeros are the μ_i. The residues w_i are > 0, and

$$\sum_{i=1}^{n} w_i = 1.$$

The function f is decreasing form $+\infty$ to $-\infty$ in each interval $]\lambda_i, \lambda_{i+1}[$. Hence f vanishes at one and only one point μ_i in $]\lambda_i, \lambda_{i+1}[$.



Consider the map

$$\begin{aligned} \Lambda^{(n)} : \quad & Herm(n, \mathbb{C}) \to (\mathbb{R}^n)_+ = \{t \in \mathbb{R}^n \mid t_1 \leq \cdots \leq t_n\} \\ & X \mapsto (\lambda_1, \ldots, \lambda_n) \end{aligned}$$

One proves more precisely

$$\Lambda^{(n)}(\mathcal{O}_A) = \{\mu \in (\mathbb{R}^{n-1})_+ \mid \mu \preceq a\}.$$

Orbital measure

The orbit $\mathcal{O}_A$ carries a natural probability measure, the orbital measure μ_A, image under the map $u \mapsto uAu^*$ of the normalized Haar measure α_n of the compact group $U(n)$.

$$\int_{Herm(n,\mathbb{C})} f(X)\mu_A(dX) = \int_{U(n)} f(uAu^*)\alpha_n(du).$$

We will describe the projection $p(\mu_A)$ of this orbital measure.

Radial part of an invariant measure

Let μ be a measure on the space $Herm(n, \mathbb{C})$ which is invariant under the transformations $X \mapsto uXu^*$ $(u \in U(n))$. Such a measure μ can be written

$$\int_{Herm(n,\mathbb{C})} f(X)\mu(dX) = \\ = \int_{(\mathbb{R}^n)_+} \Big(\int_{U(n)} f(u\mathrm{diag}(t_1, \ldots, t_n)u^*)\alpha_n(du) \Big) \nu(dt).$$

$((\mathbb{R}^n)_+ = \{t \in \mathbb{R}^n \mid t_1 \leq \cdots \leq t_n\})$.
The measure ν is called the radial part of μ.

Baryshnikov formula

Let ν_A be the radial part of the projection $p(\mu_A)$.

$$\int_{(\mathbb{R}^{n-1})_+} f(t)\nu_A(dt) = \frac{(n-1)!}{V_n(a)} \int_{a_1}^{a_2} dt_1 \int_{a_2}^{a_3} dt_2 \dots \int_{a_{n-1}}^{a_n} dt_{n-1} V_{n-1}(t) f(t).$$

$V_n(a)$ is the Vandermonde polynomial:

$$V_n(a) = \prod_{i<j} (a_j - a_i).$$

Observe that, according to the theorem of Rayleigh,

$$Support(\nu_A) = \{t \in \mathbb{R}^{n-1} \mid t \preceq a\}.$$

Sketch of the proof by Olshanski

Fourier-Laplace transform of a bounded measure μ on $Herm(n, \mathbb{C})$: for $Z \in Herm(n, \mathbb{C})$,

$$\hat{\mu}(Z) = \int_{Herm(n,\mathbb{C})} e^{\mathrm{tr}(ZX)} \mu(dX).$$

If μ is $U(n)$-invariant, then $\hat{\mu}$ is $U(n)$-invariant,
and only depends on the eigenvalues $z_1, \ldots, z_n$ of Z.
The Fourier-Laplace transform of the projection $p(\mu)$ onto $Herm(n-1, \mathbb{C})$ is equal to the restriction to $Herm(n-1, \mathbb{C})$ of the Fourier-Laplace transform of μ.

Fourier-Laplace transform of the orbital measure μ_A:

$$\begin{aligned}\widehat{\mu_A}(Z) \qquad &= \int_{Herm(n,\mathbb{C})} e^{\operatorname{tr} ZX} \mu'_A dX) \\ &= \int_{U(n)} e^{\operatorname{tr}(ZuAu^*)} \alpha_n(du).\end{aligned}$$

(α_n is the normalized Haar measure on $U(n)$.)
There is an explicit formula:

Harish-Chandra-Itzykson-Zuber integral

For $Z = \operatorname{diag}(z_1, \ldots, z_n)$,

$$\widehat{\mu_A}(Z) = \delta_n! \frac{1}{V_n(a) V_n(z)} \det(e^{a_i z_j})_{1 \le i,j \le n}.$$

$$\delta_n = (n-1, n-2, \ldots, 1, 0), \quad \delta_n! = (n-1)!(n-2)! \ldots 1!$$

Restrict to $Herm(n-1,\mathbb{C})$ amounts to taking $z_n = 0$: for $Z = \mathrm{diag}(z_1, \ldots, z_{n-1}, 0)$,

$$\widehat{\mu_A}(Z) = (-1)^{n-1}(n-1)! \qquad \frac{\delta_{n-1}!}{V_n(a)V_{n-1}(z_1,\ldots,z_{n-1})}\frac{1}{z_1\ldots z_{n-1}} \times \begin{vmatrix} e^{a_1 z_1} & \ldots & e^{a_1 z_{n-1}} & 1 \\ e^{a_2 z_1} & \ldots & e^{a_2 z_{n-1}} & 1 \\ \vdots & & \vdots & \vdots \\ e^{a_n z_1} & \ldots & e^{a_n z_{n-1}} & 1 \end{vmatrix}.$$

The result is obtained by some determinant transformations.

More generally, for $1 \leq k < n$, consider the projection p_k^n which maps $Herm(n, \mathbb{C})$ to $Herm(k, \mathbb{C})$.
A formula has been obtained by Olshanski for the radial part $\nu_A^{(k)}$ of the projection $p_k^n(\mu_A)$.
This a determinental formula involving spline functions.

It is possible to prove Olshanski 's determinantal formula by using the same scheme:

The Fourier-Laplace transform
of the projection $\mu_A^{(k)}$ is equal to the restriction of the Fourier-Laplace transform of μ_A to the subspace $Herm(k, \mathbb{C})$.

Restriction of functions defined by determinantal formulas

Let $f_1, \dots, f_n$ be n analytic functions defined in a neighborhood of 0 in $\mathbb{C}$, and F the function defined by

$$F(z_1, \dots, z_n) = \frac{1}{V_n(z)} \det\big(f_j(z_i)\big)_{1 \leq i,j \leq n}.$$

The function F, which is defined for $z_i \neq z_j$, extends as an analytic function in a neighborhood of 0 in $\mathbb{C}^n$.
The Harish-Chandra-Itzykson-Zuber formula involves a function of this type, with

$$f_j(z) = e^{a_j z}.$$

The following result is technical, but it is the key of our proof.

Theorem
For $0 \leq k \leq n-1$,

$$F(z_1, \ldots, z_k, 0, \ldots, 0) = C(n,k) \frac{1}{V_k(z_1, \ldots, z_k)} \frac{1}{(z_1 \ldots z_k)^{n-k}} \times \begin{vmatrix} f_1(z_1) & \ldots & f_n(z_1) \\ \vdots & & \vdots \\ f_1(z_k) & \ldots & f_n(z_k) \\ f_1^{(n-k-1)}(0) & \ldots & f_n^{(n-k-1)}(0) \\ \vdots & & \vdots \\ f_1'(0) & \ldots & f_n'(0) \\ f_1(0) & \ldots & f_n(0) \end{vmatrix}.$$

In particular, for $k = 1$, $f_i(z) = e^{a_i z}$,

$$F(z_1, 0, \dots, 0) = C(n,1)\frac{1}{z_1^{n-1}}\begin{vmatrix} e^{a_1 z_1} & \dots & e^{a_n z_1} \\ a_1^{n-2} & \dots & a_n^{n-2} \\ \vdots & & \vdots \\ a_1 & \dots & a_n \\ 1 & \dots & 1 \end{vmatrix}.$$

Splines

Given n knots $a_1 < a_2 < \cdots < a_n$ there is a unique function f on $\mathbb{R}$ such that

- $f(t) \geq 0$, $\mathrm{supp}(f) = [a_1, a_n]$,

- f is of class $\mathcal{C}^{n-3}$,

- The restriction of f to each interval $[a_i, a_{i+1}]$ is a polynomial of degree $\leq n-2$.

- $\int_{\mathbb{R}} f(t)dt = 1$.

This function is denoted by $M_n(a_1, \ldots, a_n; t)$, and called fundamental spline function (or B-spline). The numbers are called the knots of the spline function.

From the classical Hermite-Genocchi formula one obtains a formula for the Fourier-Laplace transform of the spline function

$$\begin{aligned}\widehat{M_n}(a_1,\ldots,a_n;z) &= \int_{\mathbb{R}} e^{zt} M_n(a_1,\ldots,a_n;t)\,dt \\ &= \frac{C}{V_n(a_1,\ldots,a_n)} \frac{1}{z^{n-1}} \begin{vmatrix} e^{a_1 z} & \ldots & e^{a_n z} \\ a_1^{n-2} & \ldots & a_n^{n-2} \\ \vdots & & \vdots \\ a_1 & \ldots & a_n \\ 1 & \ldots & 1 \end{vmatrix}.\end{aligned}$$

We can now give a formula for the densité of the projection $\mu_A^{(1)}$ of the orbital measure on $Herm(1, \mathbb{C}) \simeq \mathbb{R}$.

Theorem (Okounkov)

$$\mu_A^{(1)}(dt) = M_n(a_1, \dots, a_n)dt.$$

In fact we saw that both measures have the same Fourier-Laplace transform.

Observe that the knots of the spline function are the eigenvalues of the matrices in the orbit $\mathcal{O}_A$.

A determinantal formula has been obtained for the radial part $\nu_A^{(k)}$ of the projection $\mu_A^{(k)}$ of the orbital measure μ_A.

Theorem (Olshanski)

$$\nu_A^{(k)}(dt) = \frac{C(n,k)}{\prod_{j-i\geq n-k+1}(a_j - a_i)}$$
$$\det\big(M_{n-k+1}(a_j, \dots, a_{j+n-k}; t_i)\big)_{1\leq i,j\leq k} V_k(t)dt_1 \dots dt_k.$$

It can be proven by computing the Fourier-Laplace transform of both sides. The original proof by Olshanski is slightly different.

Olshanski's formula generalizes in the following setting: Let U be a compact simple Lie group, acting on its Lie algebra $\mathfrak{u}$ by the adjoint representation. There is an explicit formula, due to Harish-Chandra, for the Fourier-Laplace transform of an orbital measure:

Harish-Chandra formula

$$\int_U e^{\langle \mathrm{Ad}(u)H, Z\rangle} du = C \frac{\sum_{w\in W} \det w \; e^{\langle wH, Z\rangle}}{\pi(H)\pi(Z)},$$

where

$$\pi(H) = \prod_{\alpha\in R^+} \langle \alpha, H\rangle.$$

H and Z belongs to a Cartan subalgebra,
R^+ is a set of positive roots,
W is the Weyl group.

Projections of orbital measures have been considered by Zubov (2015) in this setting.
In cases of the orthogonal group SO(n) and the symplectic group $Sp(n)$, one has to determine restrictions of functions of the following type

$$D_n(f,x,y) = \frac{\det\big(f(x_i y_j)\big)}{V_n(x^2)V_n(y^2)},$$

for an even analytic function f.

Zubov has obtained determinantal formulas for the projections of orbital measures.

As in the case of the action of the unitary group $U(n)$ on the space $Herm(n,\mathbb{C})$, these formulas involve spline functions.

In this setting determinantal processes have been studied by M. Defosseux. She has obtained an analogue of the Baryshnikov formula (2010).

For the action of the orthogonal group on the space of symmetric matrices, much less is known. An explicit formula for the projection $\mu_A^{(1)}$ has been obtained by F. Fourati (2011).

Non compact analogues

We consider the action of the pseudo-orthogonal group $U(p,q)$ on the the space $Herm(n,\mathbb{C})$ $(p+q=n)$ by the transformations $X\mapsto uXu^*$.
This action is equivalent to the adjoint action of $U(p,q)$ on its Lie algebra.
Let $\Omega_n\subset Herm(n,\mathbb{C})$ be the cone of positive definite Hermitian matrices.
We will consider orbits contained in Ω_n.
For $X\in Herm(n,\mathbb{C})$, a number $\lambda\in\mathbb{C}$ will be said to be a pseudo eigenvalue if there exists a nonzero vector $v\in\mathbb{C}^n$ such that

$$Xv=\lambda I_{p,q}v,\quad I_{p,q}=\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}.$$

(Or, equivalently, λ is an eigenvalue of $I_{p,q}X$.)

For $X \in \Omega_n$, the pseudo eigenvalues are real, p pseudo eigenvalues are positive, and q are negative.

Consider a diagonal matrix $A \in \Omega_n$ with pseudo eigenvalues $a_1, \ldots, a_n$,

$$a_1 > 0, \ldots, a_p > 0, \quad a_{p+1} < 0, \ldots, a_n < 0,$$

$$A = \mathrm{diag}(a_1, \ldots, a_p, -a_{p+1}, \ldots, -a_n).$$

Then the orbit

$$\mathcal{O}_A = \{X = uAu^* \mid u \in U(p,q)\}$$

is determined by

$$\mathcal{O}_A = \big\{X \in \Omega_n \mid \text{pseudo spectrum}(X) = \{a_1, \ldots, a_n\}\big\}.$$

There is an analogue of the theorem of Rayleigh.

Assume $q \geq 1$.
For $X \in \Omega_n$ consider the projection $Y = p(X)$ of X on $Herm(n-1, \mathbb{C})$.
Pseudo eigenvalues of X:

$$\lambda_1 > \cdots > \lambda_p > 0 > \lambda_{p+1} > \cdots > \lambda_n.$$

Pseudo eigenvalues of Y:

$$\mu_1 > \cdots > \mu_p > 0 > \mu_{p+1} > \cdots > \mu_{n-1}.$$

Proposition

The pseudo eigenvalues of Y interlace the eigenvalues of X in the following way:

$$\begin{aligned} &\mu_1 > \lambda_1 > \mu_2 > \cdots \lambda_{p-1} > \lambda_p > 0 > \cdots \\ &> \lambda_{p+1} > \mu_{p+1} > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n. \end{aligned}$$

Sketch of the proof

By computing the rational function

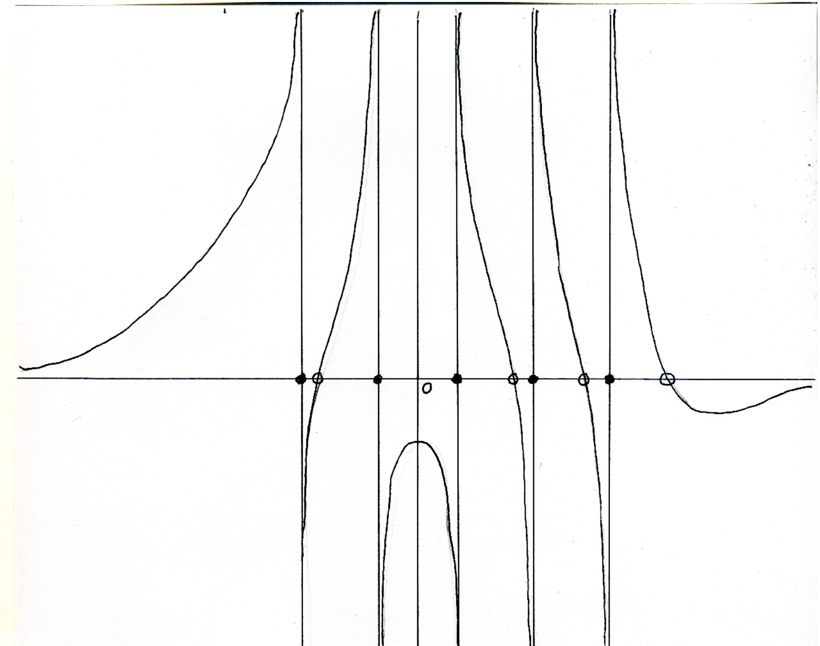
$$f(z) = \big((zI_{p,q} - X)^{-1} e_n \mid e_n\big)$$

in two ways one obtains

$$-\frac{\prod_{i=1}^{n-1}(z-\mu_i)}{\prod_{i=1}^{n}(z-\lambda_i)} = \sum_{i=1}^{n} \frac{w_i}{z-\lambda_i}.$$

Poles : pseudo eigenvalues of Y
Zeros : pseudo eigenvalues of the zeros.
Residues : $w_i > 0$ for $i = 1, \ldots, p$, $w_i < 0$ for $i = p+1, \ldots, n$, and

$$\sum_{i=1}^{n} w_i = -1.$$


0

There is also an analogue of the formula of Baryshnikov.

The orbit $\mathcal{O}_A$ carries an unbounded positive measure which is $U(p,q)$-invariant, unique up to a positive factor. Let μ_A denote such a measure. This orbital measure is the image of a Haar measure α on $U(p,q)$.

$$\int_{\mathcal{O}_A} f(X)\mu_A(dX) \int_{U(p,q)} f(uAu^*)\alpha(du).$$

Proposition The pseudo radial part ν of the projection of the orbital measure μ_A on $Herm(n-1,\mathbb{C})$ is given by

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} f(t)\nu(dt) \quad &= \frac{C}{V_n(a_1,\ldots,a_n)} \int_{a_1}^{\infty} dt_1 \int_{a_2}^{a_1} dt_2 \ldots \int_{a_p}^{a_{p-1}} dt_p \\ &\int_{a_{p+2}}^{a_{p+1}} dt_{p+1} \ldots \int_{a_n}^{a_{n-1}} dt_{n-1} V_{n-1}(t) f(t).\end{aligned}$$

The proposition can be proven by using an analogue of the Harish-Chandra-Itzykson-Zuber integral, i.e. an explicit formula for the Fourier-Laplace transform of the orbital measure μ_A: for $Z = \mathrm{diag}(z_1, \dots, z_n)$ with $\mathrm{Re}\, z_i < 0$ for $i = 1, \dots, p$, $\mathrm{Re}\, z_i > 0$ for $i = p+1, \dots, n$,

$$\int_{\mathcal{O}_A} e^{\mathrm{tr}\, ZX} \mu_A(dX) = C \frac{1}{V_n(a) V_n(z)} \det(e^{a_i z_j})_{1 \leq i,j \leq p} \det(e^{a_i z_j})_{p+1 \leq i,j \leq n}.$$

This is a special case of a formula obtained by Ben Saïd and Ørsted for reductive groups G such $G^{\mathbb{C}}/G$ is an ordered symmetric space (2005).

Baryshnikov, Yu. (2001) GUEs and queues, *Probab. Theory Relat. Fields*, 119, 256-274

Olshanski, G. & Vershik, A. (1996) Ergodic unitarily invariant measures on the space of infinite Hermitian matrices, *Amer. Math. Soc. Transl.*, 175, 137-175.

Olshanski, G. (2013). Projections of orbital measures, Gelfand-Tsetlin polytopes, and splines, *Journal of Lie Theory*, 23, 1011-1022.

Faraut, J. (2015) Rayleigh theorem, projection of orbital measures, and spline functions, *Advances in Pure and Applied Mathematics*, 6, 261-283.

Zubov, D. (2015) Projections of orbital measures associated with classical Lie groups, *preprint*.

M. Defosseux (2010) Orbit measures, random matrix theory and interlaced determinantal processes. *Ann. Inst. H. Poincaré, Probabilités et Statistiques*, 46, 209-249.

F. Fourati (2011) Dirichlet distributions and orbital integrals, *Journal of Lie theory*, 21, 189-203.

Faraut, J. (1985) Algèbres de Volterra et transformation de Laplace sphérique sur certains espaces symétriques ordonné. *Symposia Mathematica*, vol. XXIX, 183-196.

Ben Saïd, S. & B. Ørsted (2005) Bessel functions for root systems via the trigonometric setting. *Int. Math. Res. Not.* 9, 551-585.