On the Fokas-Gel'fand theorem for integrable systems

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Abstract

The Fokas-Gel'fand theorem on the immersion formula of 2D-surfaces is related to the study of Lie symmetries of an integrable system. A rigorous proof of this theorem is presented which may help to better understand the immersion formula of 2D-surfaces in Lie algebras. It is shown, that even under weaker conditions, the main results of this theorem is still valid. A connection is established between three different analytic descriptions for immersion functions of 2D-surfaces, corresponding to the following three types of symmetries: gauge symmetries of the linear spectral problem, conformal transformations in the spectral parameter and generalized symmetries of the integrable system. The theoretical results are applied to the $\mathbb{C}P^{N-1}$ sigma model and several soliton surfaces associated with these symmetries are constructed. It is shown that these surfaces are linked by gauge transformations. The Fokas-Gel'fand procedure can also be adapted for constructing soliton surfaces associated with integrable ODE's admitting Lax representations, and applied to ODE's for the elliptic and Painlevé P1 equations.

Outline

Immersion formulas for soliton surfaces

Let us consider an integrable system of PDEs in two independent variables x_1, x_2

$$\Omega[u] = 0, \tag{1}$$

where $[u] = (x, u^{(n)}) \in J^n(X \times U)$. Suppose that the system (1) is obtained as the compatibility of a matrix LSP written in the form

$$\partial_\alpha \Phi(x_1, x_2, \lambda) - U_\alpha([u], \lambda)\Phi(x_1, x_2, \lambda) = 0, \qquad \alpha = 1, 2 \tag{2}$$

In what follows, we assume that the potential matrices U_α and the wavefunction Φ can be defined on the extended jet space $\mathcal{N} = (J^n, \lambda)$, where λ is the spectral parameter. The compatibility condition of the LSP (2), often called the ZCC

$$D_2 U_1 - D_1 U_2 + [U_1, U_2] = 0, \qquad D_\alpha = \partial_\alpha + u^k_{J,\alpha}\partial_{u^k_J}, \qquad \alpha = 1, 2 \tag{3}$$

which is assumed to be valid for all values of λ, implies (1). The LSP (2) can be written as

$$D_\alpha \Phi([u], \lambda) - U_\alpha([u], \lambda)\Phi([u], \lambda) = 0. \qquad \alpha = 1, 2 \tag{4}$$

As long as the potential matrices $U_\alpha([u], \lambda)$ satisfy the ZCC (3), there exists a group-valued function Φ which satisfies (4) and consequently can be defined formally on the extended jet space $\mathcal{N} = (J^n, \lambda)$.

Immersion formulas for soliton surfaces

A. Fokas and I. Gel'fand [1996] looked for a simultaneous infinitesimal deformation of the LSP (4) which preserved ZCC (3)

$$\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{\Phi} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ \Phi \end{pmatrix} + \epsilon \begin{pmatrix} A_1 \\ A_2 \\ \Phi F \end{pmatrix} + O(\epsilon^2), \qquad 0 < \epsilon \ll 1 \tag{5}$$

where the matrices $\tilde{U}_1$, $\tilde{U}_2$, A_1, A_2 and F take values in the Lie algebra $\mathfrak{g}$, while $\tilde{\Phi} = \Phi(I + \epsilon F)$ belong to the corresponding Lie group G. The infinitesimal deformation of the ZCC (3) requires that the matrix functions A_1 and A_2 satisfy

$$D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0. \tag{6}$$

The infinitesimal deformation of the LSP (4) implies that the function F satisfies

$$D_\alpha F = \Phi^{-1} A_\alpha \Phi, \qquad \alpha = 1, 2 \tag{7}$$

The requirement (6) coincides with the compatibility condition for (7). Fokas and Gel'fand determined that the necessary condition for the existence of a $\mathfrak{g}$-valued immersion function F of a 2D-surface in $\mathfrak{g}$ can be expressed in terms of the matrices U_α and A_α which satisfy IDZCC (6).

Theorem 1 (Fokas and Gel'fand [1996])

If the matrix functions $U_\alpha \in \mathfrak{g}$, $\alpha = 1, 2$ and $\Phi \in G$ of the LSP (4) satisfy the ZCC (3) and $A_\alpha \in \mathfrak{g}$ are two linearly independent matrix functions which satisfy the IDZCC,

$$D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0, \tag{8}$$

then there exists (up to affine transformations) a 2D-surface with a $\mathfrak{g}$-valued immersion function $F([u], \lambda)$ such that the tangent vectors to this surface are linearly independent and are given by

$$D_\alpha F([u], \lambda) = \Phi^{-1} A_\alpha([u], \lambda)\Phi, \qquad \alpha = 1, 2 \tag{9}$$

The 1^{st} and 2^{nd} fundamental forms of the surface are expressible in terms of U_α, A_α only. The term integrable surfaces refers to surfaces associated with integrable GMC eqs.

Theorem 2 (A Fokas et al [2000], our formulation)

(The main result on the immersion of 2D-surfaces in Lie algebras)

Let the set of scalar functions $\{u^k\}$ satisfy an integrable system of PDEs $\Omega[u]=0$. Let the G-valued function $\Phi([u],\lambda)$ satisfy the LSP (4) of $\mathfrak{g}$-valued potentials $U_\alpha([u],\lambda)$. Let us define two linearly independent $\mathfrak{g}$-valued matrix functions $A_\alpha([u],\lambda)$ by the equations

$$A_\alpha([u],\lambda)=\beta(\lambda)D_\lambda U_\alpha+(D_\alpha S+[S,U_\alpha])+\mathrm{pr}\omega_R U_\alpha. \quad D_\lambda=\partial_\lambda \quad \alpha=1,2 \quad (10)$$

Here $\beta(\lambda)$ is an arbitrary scalar function of λ, $S=S([u],\lambda)$ is an arbitrary $\mathfrak{g}$-valued matrix function defined on the jet space $\mathcal{N}$, $\omega_R=R^k[u]\partial_{u^k}$ is the vector field, written in evolutionary form, of the generalized symmetries of the integrable PDEs $\Omega[u]=0$ given by the ZCC (3). Then there exists a 2D-surface with immersion function $F([u],\lambda)$ in the Lie algebra $\mathfrak{g}$ given by the formula (up to an additive $\mathfrak{g}$-valued constant)

$$F([u],\lambda)=\Phi^{-1}\left(\beta(\lambda)D_\lambda\Phi+S\Phi+\mathrm{pr}\omega_R\Phi\right), \quad \text{where } \omega_R=R^k[u]\partial_{u^k}. \quad (11)$$

Links between the Fréchet derivatives and evolutionary vectors fields are

$$\mathrm{pr}\omega_R U_\alpha=\frac{DU_\alpha}{Du^j}R^j, \qquad \mathrm{pr}\omega_R\Phi=\frac{D\Phi}{Du^j}R^j, \qquad \mathrm{pr}\omega_R=\omega_R+D_J R^k\partial_{u^k_J}. \quad (12)$$

Theorem 2 (A Fokas et al [2000], our formulation)

(The main result on the immersion of 2D-surfaces in Lie algebras)

$$F([u],\lambda) = \Phi^{-1}\left(\beta(\lambda) D_\lambda \Phi + S\Phi + \mathrm{pr}\omega_R \Phi\right).$$

The integrated form of the surface defines a mapping $F : \mathcal{N} \to \mathfrak{g}$ and we will refer to it as the ST immersion formula (when $S = 0, \omega_R = 0$)

$$F^{ST}([u],\lambda) = \beta(\lambda)\Phi^{-1}(D_\lambda \Phi) \in \mathfrak{g}, \tag{13}$$

the CD immersion formula (when $\beta = \omega_R = 0$)

$$F^{CD}([u],\lambda) = \Phi^{-1} S([u],\lambda)\Phi \in \mathfrak{g}, \tag{14}$$

or the FG immersion formula (when $\beta = 0, S = 0$)

$$F^{FG}([u],\lambda) = \Phi^{-1}(\mathrm{pr}\omega_R \Phi) \in \mathfrak{g}. \tag{15}$$

Immersion formulas for soliton surfaces

Let us consider the case when $\beta = S = 0$. The FG immersion function associated with the generalized symmetries of the integrable PDEs $\Omega[u] = 0$ is

$$\begin{aligned} &F([u],\lambda) = \Phi^{-1}\tfrac{D\Phi}{Du^j}R^j = \Phi^{-1}(\mathrm{pr}\omega_R\Phi) \in \mathfrak{g}, \qquad \alpha = 1,2 \\ &D_\alpha F([u],\lambda) = \Phi^{-1}A_\alpha([u],\lambda)\Phi, \qquad A_\alpha([u],\lambda) = \tfrac{DU_\alpha}{Du^j}R^j = \mathrm{pr}\omega_R U_\alpha \in \mathfrak{g}, \end{aligned} \tag{16}$$

Let us discuss the validity of the FGFI (16). A vector field written in evolutionary form ω_R defined on the jet space N

$$\omega_R = R^k[u]\frac{\partial}{\partial u^k}, \qquad \mathrm{pr}\omega_R = R^k[u]\frac{\partial}{\partial u^k} + \left(D_J R^k[u]\right)\frac{\partial}{\partial u^k_J}$$

is a generalized symmetry of the ZCC (3) iff

$$\begin{aligned} \mathrm{pr}\omega_R(D_2U_1 - D_1U_2 + [U_1,U_2]) = D_2(&\mathrm{pr}\omega_R U_1) - D_1(\mathrm{pr}\omega_R U_2) \\ &+[\mathrm{pr}\omega_R U_1, U_2] + [U_1, \mathrm{pr}\omega_R U_2] = 0 \end{aligned} \tag{17}$$

whenever $\Omega[u] = D_2U_1 - D_1U_2 + [U_1,U_2] = 0$. The expression (17) is equivalent to the IDZCC

$$D_2A_1 - D_1A_2 + [A_1,U_2] + [U_1,A_2] = 0, \qquad A_\alpha = \mathrm{pr}\omega_R U_\alpha, \qquad \alpha = 1,2 \tag{18}$$

since

$$[D_\alpha, \mathrm{pr}\omega_R] = 0, \qquad \mathrm{pr}\omega_R = R^k[u]\frac{\partial}{\partial u^k} + \left(D_J R^k[u]\right)\frac{\partial}{\partial u^k_J} \qquad \alpha = 1,2 \tag{19}$$

The Fréchet derivative of Φ with respect to u^k in the direction of R^k can be expressed through the prolongation of ω_R, i.e. $(D\Phi/Du^k)R^k = \mathrm{pr}\omega_R\Phi$. Hence

$$F = \Phi^{-1}\frac{D\Phi}{Du^k}R^k = \Phi^{-1}(\mathrm{pr}\omega_R\Phi) \tag{20}$$

Differentiating (20) and using the LSP (4) we get

$$D_\alpha F = D_\alpha\left(\Phi^{-1}\frac{D\Phi}{Du^k}R^k\right) = \Phi^{-1}\left[-U_\alpha\frac{D\Phi}{Du^k}R^k + D_\alpha\left(\frac{D\Phi}{Du^k}R^k\right)\right] \tag{21}$$

Immersion formulas for soliton surfaces

Making use of the relations (19) and (20), we can write the 2^{nd} term in (21) as

$$D_\alpha\left(\frac{D\Phi}{Du^k}R^k\right) = D_\alpha(\mathrm{pr}\omega_R\Phi) = \mathrm{pr}\omega_R(D_\alpha\Phi) \tag{22}$$

Using the identity

$$\mathrm{pr}\omega_R(D_\alpha\Phi) = \mathrm{pr}\omega_R(U_\alpha\Phi) + \mathrm{pr}\omega_R(D_\alpha\Phi - U_\alpha\Phi) \tag{23}$$

we determine that the 2^{nd} term in (23) is not necessarily zero.
This term vanishes iff the vector field ω_R is also a symmetry of the LSP (4) in the sense that

$$\mathrm{pr}\omega_R\,(D_\alpha\Phi - U_\alpha\Phi) = 0, \qquad \text{whenever} \qquad D_\alpha\Phi - U_\alpha\Phi = 0. \tag{24}$$

Let us assume that (24) holds. Then from (22) we get

$$\begin{aligned}\mathrm{pr}\omega_R(D_\alpha\Phi) &= \mathrm{pr}\omega_R(U_\alpha\Phi) = (\mathrm{pr}\omega_R U_\alpha)\Phi + U_\alpha(\mathrm{pr}\omega_R\Phi)\\ &= \left(\frac{DU_\alpha}{Du^k}R^k\right)\Phi + U_\alpha\left(\frac{D\Phi}{Du^k}R^k\right).\end{aligned} \tag{25}$$

Immersion formulas for soliton surfaces

Substituting (25) into (21) and using (18) we obtain the tangent vectors $D_\alpha F$ in the form postulated by Theorem 1

$$\begin{aligned} D_\alpha F &= \Phi^{-1}\left[-U_\alpha\left(\frac{D\Phi}{Du^k}R^k\right)+\left(\frac{DU_\alpha}{Du^k}R^k\right)+U_\alpha\left(\frac{D\Phi}{Du^k}R^k\right)\Phi\right] \\ &= \Phi^{-1}\left(\frac{DU_\alpha}{Du^k}R^k\right)\Phi = \Phi^{-1}(\mathrm{pr}\omega_R U_\alpha)\Phi = \Phi^{-1}A_\alpha([u],\lambda)\Phi \end{aligned} \tag{26}$$

Thus, under the condition that ω_R is also a symmetry of the LSP (4), there exists a 2D-surface with $\mathfrak{g}$-valued immersion function given by

$$F = \Phi^{-1}(\mathrm{pr}\omega_R\Phi) \in \mathfrak{g}. \tag{27}$$

Hence the FG immersion formula is applicable in its original form.

Immersion formulas for soliton surfaces

Proposition 1: If the vector field ω_R is a generalized symmetry of the ZCC associated with $\Omega[u] = 0$ and if two linearly independent $\mathfrak{g}$-valued matrix functions are defined by the equations

$$A_\alpha = \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R(D_\alpha \Phi - U_\alpha \Phi))\Phi^{-1}, \qquad \alpha = 1, 2, \tag{28}$$

then there exists an immersion function F of a 2D-surface which is governed by the formula (up to an additive $\mathfrak{g}$-valued constant)

$$F([u], \lambda) = \Phi^{-1}(\mathrm{pr}\omega_R \Phi) \in \mathfrak{g}, \tag{29}$$

consistent with the tangent vectors

$$D_\alpha F = \Phi^{-1}\{(\mathrm{pr}\omega_R U_\alpha)\Phi + \mathrm{pr}\omega_R(D_\alpha \Phi - U_\alpha \Phi)\}. \tag{30}$$

Proof. The IDZCC

$$D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0$$

are exactly the compatibility equation for (30) with A_α given by (28) and so the immersion function F exists and is given by (29), up to an additive $\mathfrak{g}$-valued constant. $\square$

Application of the method

The construction of soliton surfaces requires three terms for an explicit representation of the immersion function $F \in \mathfrak{g}$:

1. An LSP $D_\alpha \Phi - U_\alpha([u], \lambda)\Phi = 0$, $\alpha = 1, 2$ for the integrable PDE.
2. A generalized symmetry $\omega_R = R^k[u]\partial_{u^k}$ of the integrable PDE.
3. A solution Φ of the LSP associated with the soliton solution of the integrable PDE.

Note that item 1 is always required. In its presence, even without one of the remaining two objects, we can obtain an immersion function F.

1. When a solution Φ of the LSP is unknown, the geometry of the surface F can be obtained using the non-degenerate Killing form on the Lie algebra $\mathfrak{g}$. The 2D-surface with the immersion function F can be interpreted as a pseudo-Riemannian manifold.

2. When the generalized symmetries ω_R of the integrable PDE are unknown but we know a solution Φ of the LSP then we can define the 2D-soliton surface using the gauge transformation and the λ-invariance of the ZCC

$$F = \Phi^{-1}(\beta(\lambda)D_\lambda \Phi + S\Phi), \tag{31}$$

Application of the method

Equation (31) is consistent with the tangent vectors

$$D_\alpha F = \beta(\lambda)\Phi^{-1}(D_\lambda U_\alpha) + \Phi^{-1}(D_\alpha S + [S, U_\alpha])\Phi. \tag{32}$$

In all cases, the tangent vectors and the unit normal vector to a 2D-surface expressed in terms of matrices $A_1, A_2 \in \mathfrak{g}$ are

$$D_\alpha F = \Phi^{-1} A_\alpha \Phi \in \mathfrak{g}, \qquad N = \frac{\Phi^{-1}[A_1, A_2]\Phi}{(\frac{\epsilon}{2}\mathrm{tr}[A_1, A_2]^2)^{1/2}} \in \mathfrak{g}, \qquad \epsilon = \pm 1 \tag{33}$$

$$A_\alpha = \beta(\lambda)(D_\lambda U_\alpha) + (D_\alpha S + [S, U_\alpha]) + \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R(D_\alpha \Phi - U_\alpha \Phi))\Phi^{-1}. \quad \alpha = 1, 2 \tag{34}$$

The first and second fundamental forms are given by

$$I = g_{ij} dx_i dx_j, \qquad II = b_{ij} dx_i dx_j, \qquad i = 1, 2 \tag{35}$$

where

$$g_{ij} = \frac{\epsilon}{2}\epsilon \mathrm{tr}(A_i A_j) \qquad b_{ij} = \frac{\epsilon}{2}\mathrm{tr}((D_j A_i + [A_i, U_j])N), \quad \epsilon = \pm 1. \tag{36}$$

Application of the method

This gives the following expressions for the mean and Gaussian curvatures

$$
\begin{aligned}
H &= \frac{1}{\Delta}\left\{\mathrm{tr}(A_2^2)\mathrm{tr}((D_1A_1+[A_1,U_1])N)\right. \\
&\quad -8\mathrm{tr}(A_1A_2)\mathrm{tr}((D_2A_1+[A_1,U_2])N) \\
&\quad \left.+\mathrm{tr}(A_1^2)\mathrm{tr}((D_2A_2+[A_2,U_2])N)\right\}, \\
K &= \frac{1}{\Delta}\{\mathrm{tr}((D_1A_1+[A_1,U_1])N) \\
&\quad \cdot\mathrm{tr}((D_2A_2+[A_2,U_2])N) \\
&\quad -2\mathrm{tr}^2((D_2A_1+[A_1,U_2])N)\}, \\
\Delta &= \mathrm{tr}(A_1^2)\mathrm{tr}(A_2^2)-4\mathrm{tr}(A_1A_2),
\end{aligned}
\tag{37}
$$

which are expressible in terms of U_α and A_α only.

Conformal symmetries and gauge transformations

Proposition 2: A symmetry of the ZCC (3) of the LSP associated with an integrable system $\Omega[u]=0$ is a λ-conformal symmetry iff there exists a $\mathfrak{g}$-valued matrix function (gauge) $S_1 = S_1([u],\lambda)$ which is a solution of the system of PDEs

$$D_\alpha S_1 + [S_1, U_\alpha] = \beta(\lambda) D_\lambda U_\alpha. \qquad \alpha = 1,2 \tag{38}$$

Outline of the proof. ($\Rightarrow$) The linearly independent matrices

$$A_\alpha([u],\lambda) = \beta(\lambda) D_\lambda U_\alpha([u],\lambda) \in \mathfrak{g}, \qquad \alpha = 1,2 \tag{39}$$

associated with the λ-conformal symmetry of the ZCC (3) satisfy the IDZCC

$$D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0 \tag{40}$$

and the corresponding ST immersion function is

$$F^{ST}([u],\lambda) = \beta(\lambda)\Phi^{-1} D_\lambda \Phi \in \mathfrak{g}, \tag{41}$$

with linearly independent tangent vectors

$$D_\alpha F^{ST} = \beta(\lambda)\Phi^{-1}(D_\lambda U_\alpha)\Phi, \qquad \alpha = 1, 2. \tag{42}$$

Any $\mathfrak{g}$-valued matrix function can be written as the adjoint group action on its Lie algebra. This implies the existence of a matrix function $S_1([u], \lambda) \in \mathfrak{g}$ for which the STIF (41) is the CDIF, i.e.

$$F^{CD}([u], \lambda) = \Phi^{-1} S_1([u], \lambda)\Phi \in \mathfrak{g}, \tag{43}$$

with tangent vectors

$$D_\alpha F^{CD} = \Phi^{-1}(D_\alpha S_1 + [S_1, U_\alpha])\Phi, \qquad \alpha = 1, 2. \tag{44}$$

By comparing the tangent vectors (42) and (44) we obtain the system of PDEs (38).The system (38) is a solvable one since

$$\begin{aligned}&\beta(\lambda)D_2(D_\lambda U_1) - \beta(\lambda)D_1(D_\lambda U_2) - [\beta(\lambda)D_\lambda U_2 - [S_1, U_2], U_1] - [S_1, D_2 U_1]\\ &+[\beta(\lambda)D_\lambda U_1 - [S_1, U_1], U_2] + [S_1, D_1 U_2] = 0,\end{aligned} \tag{45}$$

is identically satisfied whenever the ZCC (3) and the system of PDEs (38) hold.

So if we can find a gauge $S_1([u], \lambda)$ which satisfies the system of PDEs (38), then the STIF (41) can always be represented by a gauge.

$(\Leftarrow)$ Conversely, comparing the immersion formulas (41) with (43) we find a linear matrix equation for Φ

$$D_\lambda \Phi = \frac{1}{\beta(\lambda)} S_1([u], \lambda)\Phi. \tag{46}$$

If the gauge $S_1([u], \lambda)$ is known, then by solving (46) we can determine Φ and obtain the STIF for 2D-soliton surfaces. Hence, the STIF (41) is equivalent to the CDIF (43) for the gauge S_1, which satisfies the system of PDEs (38),

$$D_\alpha S_1 + [S_1, U_2] = \beta(\lambda) D_\lambda U_\alpha, \qquad \alpha = 1, 2 \qquad\qquad \square$$

Generalized symmetries and gauge transformations

Proposition 3: A vector field $\omega_R = R^k[u]\partial_{u^k}$ is a generalized symmetry of the ZCC (3) associated with $\Omega[u] = 0$ iff there exists a $\mathfrak{g}$-valued matrix function (gauge) $S_2 = S_2([u], \lambda)$ which is a solution of the system of PDEs

$$D_\alpha S_2 + [S_2, U_\alpha] = \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R(D_\alpha \Phi - U_\alpha \Phi))\,\Phi^{-1}. \qquad \alpha = 1, 2 \tag{47}$$

Outline of the proof. $(\Rightarrow)$ An evolutionary vector field ω_R is a generalized symmetry of the ZCC (3) iff

$$\mathrm{pr}\omega_R(D_2 U_1 - D_1 U_2 + [U_1, U_2]) = 0, \tag{48}$$

whenever

$$D_2 U_1 - D_1 U_2 + [U_1, U_2] = 0. \tag{49}$$

Eq (48) is equivalent to the IDZCC (6), with two linearly independent matrices

$$A_\alpha([u], \lambda) = \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R(D_\alpha \Phi - U_\alpha \Phi))\Phi^{-1} \in \mathfrak{g}, \qquad \alpha = 1, 2 \tag{50}$$

which identically satisfy the IDZCC

$$D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0.$$

Generalized symmetries and gauge transformations

An integrated form of the FGIF associated with the vector field ω_R and the tangent vectors are given by are given by

$$F^{FG}([u],\lambda) = \Phi^{-1}(\mathrm{pr}\omega_R \Phi) \in \mathfrak{g}, \qquad D_\alpha F^{FG}([u],\lambda) = \Phi^{-1} A_\alpha([u],\lambda)\Phi. \tag{51}$$

$$A_\alpha([u],\lambda) = \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R(D_\alpha \Phi - U_\alpha \Phi))\Phi^{-1}$$

Any $\mathfrak{g}$-valued matrix function can be written as the adjoint group action on its Lie algebra. This implies the existence of a matrix function $S_2([u],\lambda) \in \mathfrak{g}$, for which the FGIF (51) is the CDFI

$$F^{CD}([u],\lambda) = \Phi^{-1} S_2([u],\lambda)\Phi \in \mathfrak{g}, \tag{52}$$

with tangent vectors

$$D_\alpha F^{CD} = \Phi^{-1}(D_\alpha S_2 + [S_2, U_\alpha])\Phi, \qquad \alpha = 1, 2. \tag{53}$$

By comparing the tangent vectors (51) and (44) we obtain the system of PDEs (47). The system (47) is a solvable one, since

$$[S_2, D_2U_1 - D_1U_2] + [[S_2, U_1], U_2] - [[S_2, U_2], U_1] = 0, \tag{54}$$

which is identically satisfied whenever the ZCC (3) and (47) hold. So if one can find a gauge S_2 which satisfies (47), then the FGIF (51) can be represented by a gauge.
($\Leftarrow$) Conversely, comparing the immersion formulas (51) and (43) we find

$$\mathrm{pr}\omega_R\Phi = S_2([u], \lambda)\Phi \tag{55}$$

If the gauge S_2 is known, then by solving (55) we can determine Φ and obtain the FGIF for 2D-surfaces. Hence, the FGIF (51) is equivalent to the CDIF for the gauge S_2, which satisfies the system of PDEs (47)

$$D_\alpha S_2 + [S_2, U_\alpha] = \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R(D_\alpha\Phi - U_\alpha\Phi))\Phi^{-1}.$$

$\square$

The Sym-Tafel immersion formula versus the Fokas-Gel'fand immersion formula

Proposition 4: Suppose that the gauges S_1 and S_2 are the two $\mathfrak{g}$-valued matrix functions which are solutions of the systems of PDEs

$$\begin{aligned} D_\alpha S_1 + [S_1, U_\alpha] &= \beta(\lambda) D_\lambda U_\alpha, \qquad \alpha = 1, 2, \\ D_\alpha S_2 + [S_2, U_\alpha] &= \mathrm{pr}\omega_R U_\alpha + (\mathrm{pr}\omega_R (D_\alpha \Phi - U_\alpha \Phi))\, \Phi^{-1}, \end{aligned} \tag{56}$$

respectively.
If the gauge S_2 is a non-singular matrix then there exists a matrix $(S_1 \cdot S_2^{-1})$ which defines a mapping from the FG immersion formula (51) to the ST immersion formula (41)

$$D_\lambda \Phi = \frac{1}{\beta(\lambda)} (S_1 \cdot S_2^{-1})(\mathrm{pr}\omega_R \Phi). \tag{57}$$

Alternatively, if the gauge S_1 is a non-singular matrix, then their exists a matrix $(S_2 \cdot S_1^{-1})$ which defines a mapping from the ST immersion formula (41) to the FG immersion formula (51)

$$\mathrm{pr}\omega_R \Phi = \beta(\lambda)(S_2 \cdot S_1^{-1})(D_\lambda \Phi). \tag{58}$$

The Sym-Tafel immersion formula versus the Fokas-Gel'fand immersion formula

$$\beta(\lambda)(D_\lambda \Phi) = S_1 S_2^{-1}(\mathrm{pr}\omega_R \Phi) \tag{57}$$

Proof. Equation (57) is obtained by eliminating the wavefunction Φ from the right-hand side of equations

$$\beta(\lambda) D_\lambda \Phi = S_1 \Phi, \qquad \mathrm{pr}\omega_R \Phi = S_2 \Phi, \tag{59}$$

respectively. So the link between the immersion functions F^{ST} and F^{FG} exists, up to a $\mathfrak{g}$-valued constant gauge. $\square$

The Sym-Tafel immersion formula versus the Fokas-Gel'fand immersion formula

$$\begin{tikzcd}
& F^{ST} = \beta(\lambda)\Phi^{-1}(D_\lambda \Phi) \in \mathfrak{g} \arrow[dd, shift left, "S_2 \circ S_1^{-1}"] \\
\Phi \in G \arrow[ur, "S_1 \in \mathfrak{g}"] \arrow[dr, "S_2 \in \mathfrak{g}"'] & \\
& F^{FG} = \Phi^{-1}(\mathrm{pr}\,\omega_R \Phi) \in \mathfrak{g} \arrow[uu, shift left, "S_1 \circ S_2^{-1}"]
\end{tikzcd}$$

Figure: Representation of the relations between the wavefunction $\Phi \in G$ and the $\mathfrak{g}$-valued ST and FG formulas for immersions of 2D-soliton surfaces.

$$S_1 = \beta(\lambda)(D_\lambda \Phi)\Phi^{-1}, \qquad S_2 = (\mathrm{pr}\,\omega_R \Phi)\Phi^{-1} \tag{60}$$

To conclude, in all three cases we give explicit expressions for 2D-soliton surfaces immersed in the Lie algebra $\mathfrak{g}$ and demonstrate that one such surface can be transformed to another one through a gauge.

The $\mathbb{C}P^{N-1}$ sigma model and soliton surfaces

Consider the $\mathbb{C}P^{N-1}$ model in terms of a rank-one Hermitian projector P

$$\begin{array}{ll} [\partial_+\partial_- P, P] = \emptyset & \partial_\pm = \frac{1}{2}(\partial_1 \pm i\partial_2) \\ P^2 = P^\dagger = P, \mathrm{tr} P = 1 & \partial_1 = \partial_{\xi_1}, \partial_2 = \partial_{\xi_2} \end{array} \tag{61}$$

We assume that the model is defined on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ and that its action functional is finite. There exist raising and lowering operators $\Pi_\pm$ of solutions of (61) and any solution can be expressed as a raising operator acting on the holomorphic solution

$$\Pi_\pm(P) = \begin{cases} \dfrac{(\partial_\pm P)P(\partial_\mp P)}{\mathrm{tr}(\partial_\pm P P \partial_\mp P)} & \text{for } (\partial_\pm P)P(\partial_\mp P) \neq \emptyset \\ \emptyset & \text{for } (\partial_\pm P)P(\partial_\mp P) = \emptyset \end{cases}$$

$$\Pi_-(P_k) = P_{k-1}, \qquad \Pi_+(P_k) = P_{k+1}, \qquad \sum_{j=0}^{N-1} P_j = \mathbb{I}_N, \qquad P_k P_j = \delta_{kj} P_k$$

The generalized Weierstrass formula for immersion (GWFI) of 2D-surfaces in $\mathfrak{su}(N)$ is defined by

$$F_k(\xi, \bar{\xi}) = i \int_\gamma (-[\partial P_k, P_k] d\xi + [\bar{\partial} P_k, P_k] d\bar{\xi}) \in \mathfrak{g}, \qquad 0 \leq k \leq N-1. \tag{62}$$

The $\mathbb{C}P^{N-1}$ sigma model and soliton surfaces

The LSP is given by

$$\partial_\alpha \Phi_k = U_{\alpha k}\Phi_k, \quad U_{\alpha k} = \frac{2}{1 \pm \lambda}[\partial_\alpha P_k, P_k], \quad (U_{1k})^\dagger = -U_{2k}, \quad 0 \leq k \leq N-1 \tag{63}$$

(where $\alpha = 1, 2$ stands for $\pm$) with solution $\Phi = \Phi([P], \lambda)$ which goes to the identity matrix $\mathbb{I}_N$ as $\lambda \to \infty$

$$\Phi = \mathbb{I}_N + \frac{4\lambda}{(1-\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1-\lambda}P_k \in SU(N), \qquad \lambda = it, \quad t \in \mathbb{R} \tag{64}$$

For the surfaces corresponding to the projectors P_k, the integration of the GWFI is performed explicitly

$$F_k = -i\left(P_k + 2\sum_{j=0}^{k-1} P_j\right) + ic_k\mathbb{I}_N \in \mathfrak{su}(N), \qquad c_k = \frac{2k+1}{N}, \tag{65}$$

and satisfies the algebraic conditions

$$[F_k - ic_k\mathbb{I}_N][F_k - i(c_k-1)\mathbb{I}_N][F_k - i(c_k-2)\mathbb{I}_N] = 0, \qquad \sum_{k=0}^{N-1}(-1)^k F_k = 0. \tag{66}$$

The $\mathbb{C}P^{N-1}$ sigma model and soliton surfaces

We express the model in terms of elements of the $\mathfrak{su}(N)$ algebra instead of P_k

$$\theta_k \equiv i\left(P_k - \frac{1}{N}\mathbb{I}_N\right) \in \mathfrak{su}(N) \tag{67}$$

with algebraic restriction

$$\theta_k \cdot \theta_k = -i\frac{(2-N)}{N}\theta_k + \frac{(1-N)}{N^2}\mathbb{I}_N \Leftrightarrow P_k^2 = P_k. \tag{68}$$

The Euler-Lagrange equations become (for simplicity we drop the index k)

$$\Omega^j[\theta] = \left[(\partial_1^2 + \partial_2^2)\theta, \theta\right]^j = 0, \qquad j = 1, ..., N^2 - 1 \tag{69}$$

where $[\cdot,\cdot]^j$ denotes the coefficients of the j^{th} basis element e_j for the $\mathfrak{su}(N)$ algebra. The potential matrices U_α expressed in terms of θ are

$$\begin{aligned} U_1 &= \frac{-2}{1-\lambda^2}\left([\partial_1\theta,\theta] - i\lambda[\partial_2\theta,\theta]\right) \in \mathfrak{su}(N), \qquad \lambda = it, \qquad t \in \mathbb{R} \\ U_2 &= \frac{-2}{1-\lambda^2}\left(i\lambda[\partial_1\theta,\theta] + [\partial_2\theta,\theta]\right) \in \mathfrak{su}(N). \end{aligned} \tag{70}$$

Expressing the wavefunction Φ in terms of $\theta \in \mathfrak{su}(N)$, we get

$$\Phi([\theta], \lambda) = \mathbb{I}_N + \frac{4\lambda}{(1-\lambda)^2} \sum_{j=0}^{N} \Pi_-^j(\theta) - \frac{2}{1-\lambda}\left(\frac{1}{N}\mathbb{I}_N - i\theta\right) \in SU(N) \tag{71}$$

$$\Pi_-(\theta) = \frac{\bar{\partial}\theta(\mathcal{E} - i\theta)\partial\theta}{\mathrm{tr}(\bar{\partial}\theta(\mathcal{E} - i\theta)\partial\theta)}, \qquad \Pi_+(\theta) = \frac{\partial\theta(\mathcal{E} - i\theta)\bar{\partial}\theta}{\mathrm{tr}(\partial\theta(\mathcal{E} - i\theta)\bar{\partial}\theta)}, \qquad \mathcal{E} = \frac{1}{N}\mathbb{I}_N, \tag{72}$$

For any functions f and g, the E-L eqs (69) and its LSP (63) (with the potential matrix (70)) admit the conformal symmetries

$$\omega_{C_i} = \left[f(\xi_i)\partial_1\theta^j + g(\xi_i)\partial_2\theta^j\right]\frac{\partial}{\partial\theta^j}, \qquad i = 1, 2. \tag{73}$$

The vector fields ω_{C_i} are related to the fields

$$\eta_{C_i} = (\partial_i\Phi^j)\frac{\partial}{\partial\Phi^j} + (\partial_i U_\alpha^j)\frac{\partial}{\partial U_\alpha^j}, \qquad i = 1, 2 \tag{74}$$

which are confomal symmetries of the LSP (63). The integrated form of the surface is given by the FG formula

$$F([\theta], \lambda) = \Phi^{-1}\left(f(\xi_1)U_1 + g(\xi_2)U_2\right)\Phi \in \mathfrak{su}(N). \tag{75}$$

Soliton surfaces associated with the $\mathbb{C}P^1$ sigma model

The simplest solutions of the $\mathbb{C}P^{N-1}$ model constitute the Veronese sequence

$$f = \left(1, \begin{pmatrix} N-1 \\ 1 \end{pmatrix}^{1/2} z, ..., \begin{pmatrix} N-1 \\ r \end{pmatrix}^{1/2} z^r, ..., z^{N-1}\right), \qquad P_k = \frac{f_k \otimes f^\dagger}{f_k^\dagger f_k},$$

$$z = x + iy \in \mathbb{C}, \qquad f_{k+1} = (\mathbb{I}_N - P_k)\partial f_k, \qquad 0 \leq k \leq N-1. \tag{76}$$

The only solutions for which the action of the $\mathbb{C}P^1$ ($N = 2$) model is finite are holomorphic P_0 and antiholomorphic P_1 projectors.

$$\begin{aligned} P_0 &= \tfrac{f_0 \otimes f_0^\dagger}{f_0^\dagger f_0} = \tfrac{1}{1+|z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix}, \quad f_0 = (1, z), \qquad k = 0 \\ P_1 &= \tfrac{f_1 \otimes f_1^\dagger}{f_1^\dagger f_1} = \tfrac{1}{1+|z|^2} \begin{pmatrix} |z|^2 & -\bar{z} \\ -z & 1 \end{pmatrix}, \quad f_1 = (\mathbb{I}_2 - P_0)\partial f_0, \quad k = 1 \end{aligned} \tag{77}$$

The integrated forms of the surfaces are given by

$$\begin{aligned} F_0 &= i(\tfrac{1}{2}\mathbb{I}_2 - P_0) = \tfrac{i}{1+|z|^2} \begin{pmatrix} \frac{1}{2}(|z|^2 - 1) & -\bar{z} \\ -z & \frac{1}{2}(1 - |z|^2) \end{pmatrix} \in \mathfrak{su}(2), \\ F_1 &= -i(P_1 + 2P_0) + \tfrac{3i}{2}\mathbb{I}_2 = F_0. \end{aligned} \tag{78}$$

Soliton surfaces associated with the $\mathbb{C}P^1$ sigma model

The potential matrices $U_{\alpha k}$ become

$$\begin{aligned}
&U_{10} = U_{11} = \tfrac{2}{(\lambda+1)(1+|z|^2)^2} \begin{pmatrix} -\bar{z} & -\bar{z}^2 \\ 1 & \bar{z} \end{pmatrix}, \qquad \lambda = it, \qquad k = 0, \\
&U_{20} = U_{21} = \tfrac{2}{(\lambda-1)(1+|z|^2)^2} \begin{pmatrix} -z & 1 \\ -z^2 & z \end{pmatrix}, \qquad t \in \mathbb{R}, \qquad k = 1, \\
&(U_{1k})^\dagger = -U_{2k}.
\end{aligned} \tag{79}$$

The $SU(2)$-valued soliton wavefunction Φ_k in the LSP take the forms

$$\begin{aligned}
&\Phi_0 = \tfrac{1}{1+|z|^2} \begin{pmatrix} \frac{-i+t+(i+t)|z|^2}{t-i} & \frac{-2i\bar{z}}{t-i} \\ \frac{-2iz}{t+i} & \frac{i+t+(t-i)|z|^2}{t+i} \end{pmatrix}, \qquad k = 0, \\
&\Phi_1 = \tfrac{1}{1+|z|^2} \begin{pmatrix} \frac{1+t^2+(t+i)^2|z|^2}{(t-i)^2} & \frac{2(1-it)\bar{z}}{(t-i)^2} \\ \frac{-2i(t-i)z}{(t+i)^2} & \frac{1+t^2+(t-i)^2|z|^2}{(t+i)^2} \end{pmatrix}, \qquad k = 1.
\end{aligned} \tag{80}$$

Let us consider separately four different analytic descriptions for the immersion functions of 2D-surfaces in $\mathfrak{su}(2)$ which are related to four different types of symmetries.

1. The Sym-Tafel formula for immersion (conformal symmetry in λ)

The ZCC of the CP^1 model admits a conformal symmetry in the spectal parameter λ. The tangent vectors $D_\alpha F_k^{ST}$ associated with this symmetry are given by

$$D_\alpha F_k^{ST} = -i\Phi_k^{-1}(D_\lambda U_{\alpha k})\Phi_k, \qquad \text{where } \beta(\lambda) = i, \qquad \alpha = 1,2, \qquad k = 0,1$$

are linearly independent. The integrated forms of the 2D-surfaces in $\mathfrak{su}(2)$ are given by the ST formula

$$F_0^{ST} = -i\Phi_0^{-1}(D_\lambda \Phi_0) = \frac{-2}{(1+t^2)^2(1+|z|^2)^2}$$

$$\begin{pmatrix} -|z|^2[t^2-3+|z|^2(1+t^2)] & \bar{z}[(t+i)^2+|z|^2(3+2it+t^2)] \\ z[(t-i)^2+|z|^2(3-2it+t^2)] & |z|^2[t^2-3+|z|^2(t^2+1)] \end{pmatrix}, \qquad k = 0$$

$$F_1^{ST} = -i\Phi_1^{-1}(D_\lambda \Phi_1) = \frac{-2}{(1+t^2)^2(1+|z|^2)^2} \qquad k = 1$$

$$\begin{pmatrix} -[(t^2+1)(1+2|z|^4)+3|z|^2(t^2-3)] & \bar{z}[6it-5+t^2+|z|^2(7+6it+t^2)] \\ z[t^2-6it-5+|z|^2(7+t^2-6it)] & (t^2+1)(1+2|z|^4)+3|z|^2(t^2-3) \end{pmatrix}. \tag{81}$$

1. The Sym-Tafel formula for immersion (conformal symmetry in λ)

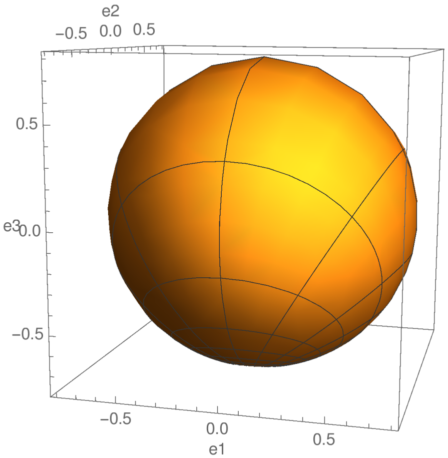
The surfaces F_k^{ST} have positive constant Gaussian and mean curvatures and are spheres (see Fig. 2a)

$$K = H = 4. \tag{82}$$

The $\mathfrak{su}(2)$-valued gauges S_k^{ST} associated with the STIF F_k^{ST} are

$$\begin{aligned}
&S_0^{ST} = (D_\lambda \Phi_0)\Phi_0^{-1} = \tfrac{-2}{1+|z|^2}\begin{pmatrix} \frac{-|z|^2}{t^2+1} & \frac{\bar{z}}{(t-i)^2} \\ \frac{z}{(t+i)^2} & \frac{|z|^2}{t^2+1} \end{pmatrix}, \qquad k = 0, \\
&S_1^{ST} = (D_\lambda \Phi_1)\Phi_1^{-1} = \tfrac{-2}{1+|z|^2}\begin{pmatrix} \frac{-(1+2|z|^2)}{t^2+1} & \frac{\bar{z}(t+i)^2}{(t-i)^4} \\ \frac{z(t-i)^2}{(t+i)^4} & \frac{1+2|z|^2}{t^2+1} \end{pmatrix}, \qquad k = 1, \\
&\det S_k^{ST} \neq 0, \qquad \mathrm{tr} S_k^{ST} = 0.
\end{aligned} \tag{83}$$

1. The Sym-Tafel formula for immersion (conformal symmetry in λ)



For $\mathbb{C}P^1$ model we have $(F_k^{ST})^2 + \frac{1}{4}\mathbb{I}_2 = 0$, $k = 0, 1$ and $F_k^{ST} = -i\sum_{\alpha=1}^{3} x_{\alpha k}\sigma_\alpha$ $\Rightarrow x_{1k}^2 + x_{2k}^2 + x_{3k}^2 = \frac{1}{4}$, spheres. (In what follows we use coordinate notation (x,y) on a surface) $F_k^{ST} = \left\{ \frac{x}{1+x^2+y^2}, \frac{y}{1+x^2+y^2}, \frac{1-x^2-y^2}{2(1+x^2+y^2)} \right\}$

2. The Fokas-Gel'fand formula for immersion (scaling symmetries)

The surfaces $F_k^g \in \mathfrak{su}(2)$ associated with the sclaing symmetries of the $\mathbb{C}P^1$ model

$$\omega_k^g = (D_1(zU_{1k}) + \bar{z}(D_2 U_{1k}))\,\tfrac{\partial}{\partial\theta^1} + (z(D_1 U_{2k}) + D_2(\bar{z} U_{2k}))\,\tfrac{\partial}{\partial\theta^2}, \tag{84}$$

have integrated form

$$F_k^g = \Phi_k^{-1}(zU_{1k} + \bar{z}U_{2k})\Phi_k, \qquad k = 0, 1 \tag{85}$$

(where $U_{\alpha k}$ is given by eqs (79))The surfaces F_k^g also have positive curvatures

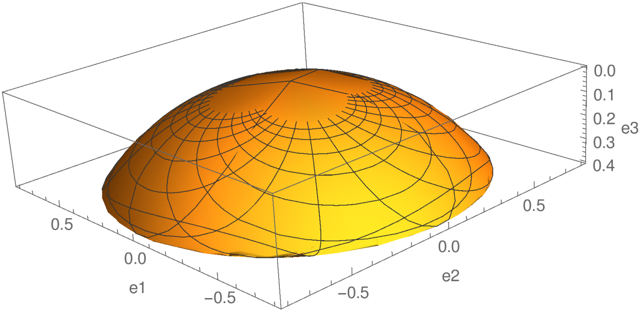
$$K_0 = K_1 = -4\lambda^2, \qquad H_0 = H_1 = -4i\lambda, \qquad i\lambda \in \mathbb{R}. \tag{86}$$

but are not spheres, since they have boundaries (see Fig. 2b). The $\mathfrak{su}(2)$-valued gauges $S_k^g = (\mathrm{pr}\omega_k^g \Phi_k)\Phi_k^{-1}$ associated with ω_k^g are given by

$$S_0^g = S_1^g = \tfrac{2}{(t^2+1)(1+|z|^2)^2}\begin{pmatrix} 2it|z|^2 & i\bar{z}[i - t + |z|^2(t+i)] \\ z[1 - it + |z|^2(1+it)] & -2it|z|^2 \end{pmatrix}, \tag{87}$$

where $\det S_k^g \neq 0$.

2. The Fokas-Gel'fand formula for immersion (scaling symmetries)



A part of ellipsoid: $F_k^g = \left(\dfrac{x^3 - 2x^2y + x(y^2-1) - 2y(1+y^2)}{(1+x^2+y^2)^2}, \right.$
$\left. -\dfrac{2x^3 + x^2y + y(y^2-1) + 2x(1+y^2)}{(1+x^2+y^2)^2}, \dfrac{2(x^2+y^2)}{(1+x^2+y^2)^2} \right)$

3. The Fokas-Gel'fand formula for immersion (conformal symmetries)

The surfaces associated with the conformal symmetry

$$\omega_k^c = -g_k(z)\partial - \bar{g}_k(\bar{z})\bar{\partial}, \qquad g_k(z) = 1 + i, \tag{88}$$

have the integrated forms

$$F_k^c = \Phi_k^{-1}(U_{1k} + U_{2k})\Phi_k, \qquad k = 0, 1 \tag{89}$$

where

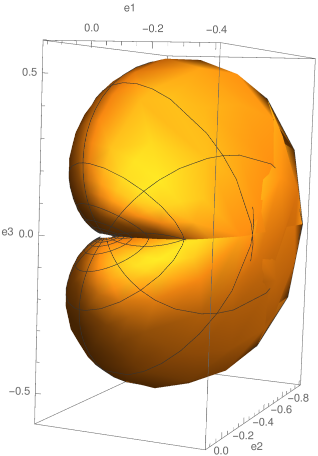
$$U_{10} + U_{20} = U_{11} + U_{21} = \tfrac{2}{(t^2+1)(1+|z|^2)^2} \begin{pmatrix} 2z + i(t+i)(z+\bar{z}) & -1 - it + i\bar{z}^2(t+i) \\ 1 + z^2 + it(z^2-1) & -[2z + i(t+i)(z+\bar{z})] \end{pmatrix}. \tag{90}$$

The surfaces F_k^c have the Euler-Poincaré characters

$$\chi_k = \frac{-1}{\pi}\int\int_{S^2} \partial\bar{\partial}\ln\left[tr(\partial P_k \cdot \bar{\partial}P_k)\right]dx^1 dx^2 = 2 \tag{91}$$

and $K > 0$ means that F_k^c are homeomorphic to ovaloids (see Fig. 2c).

3. The Fokas-Gel'fand formula for immersion (conformal symmetries)



A cardioid surface: $F_k^c = \left(\frac{x^2 - 1 - 4xy - y^2}{(1+x^2+y^2)^2}, -\frac{2(1+x^2+xy-y^2)}{(1+x^2+y^2)^2}, \frac{2(2x-y)}{(1+x^2+y^2)^2} \right)$

The $\mathfrak{su}(2)$-valued gauges S_k^c associated with ω_k^c take the form

$$S_0^c=(\mathrm{pr}\omega^c\Phi_0)\Phi_0^{-1}=\frac{2}{(1+|z|^2)^2}\begin{pmatrix}\frac{-i(1-i)(t-i)z+(1+i)(1-it)\bar{z}}{t^2+1} & \frac{(1-i)(1+it)+(1+i)(1-it)\bar{z}^2}{(t-i)^2}\\ \frac{i(1+i)(t+i)-(1-i)z^2(t-i)}{(t+i)^2} & \frac{(1-i)(1+it)z+i(1+i)(t+i)\bar{z}}{t^2+1}\end{pmatrix},\qquad k=0$$

$$S_1^c=(\mathrm{pr}\omega^c\Phi_1)\Phi_1^{-1}=\frac{2}{(1+|z|^2)^2}\begin{pmatrix}\frac{-i(1-i)(t-i)z+(1+i)(1-it)\bar{z}}{t^2+1} & \frac{(t+i)^2[(1-i)(1+it)+(1+i)(1-it)\bar{z}^2]}{(t-i)^4}\\ \frac{i(t-i)^2[(1+i)(t+i)-(1-i)(t-i)z^2]}{(t+i)^4} & \frac{(1-i)(1+it)z+i(1+i)(t+i)\bar{z}}{t^2+1}\end{pmatrix},\qquad k=1 \tag{92}$$

where $\det S_k^c\neq 0$.

4. The Fokas-Gel'fand formula for immersion (generalized symmetries)

The surfaces associated with the generalized symmetries

$$\begin{aligned}\omega_k^R &= (D_1^2 U_{1k} + D_2^2 U_{1k} + [D_1 U_{1k}, U_{1k}] + [D_2 U_{1k}, U_{1k}])\tfrac{\partial}{\partial\theta^1} \\ &+(D_1^2 U_{2k} + D_2^2 U_{2k} + [D_2 U_{2k}, U_{2k}] + [D_1 U_{2k}, U_{2k}])\tfrac{\partial}{\partial\theta^2}, \quad k = 0, 1\end{aligned} \tag{93}$$

(where $U_{\alpha k}$ is given by eqs (79)) have the integrated form

$$F_k^{FG} = \Phi^{-1}(\mathrm{pr}\omega_k^R \Phi_k) = \Phi_k^{-1}(D_1 U_{1k} + D_2 U_{2k})\Phi_k. \tag{94}$$

consistent with the tangent vectors

$$\begin{aligned}D_1 F_k^{FG} &= \Phi_k^{-1}(\mathrm{pr}\omega_k^R U_{1k})\Phi_k \\ &= \Phi_k^{-1}(D_1^2 U_{1k} + D_2^2 U_{1k} + [D_1 U_{1k}, U_{1k}] + [D_2 U_{1k}, U_{1k}])\Phi_k, \\ D_2 F_k^{FG} &= \Phi_k^{-1}(\mathrm{pr}\omega_k^R U_{2k})\Phi_k \\ &= \Phi_k^{-1}(D_1^2 U_{2k} + D_2^2 U_{2k} + [D_2 U_{2k}, U_{2k}] + [D_1 U_{2k}, U_{2k}])\Phi_k.\end{aligned} \tag{95}$$

The surfaces F_k^{FG} have $K > 0$ and $\chi = 2$ and they are homeomorphic to ovaloids.

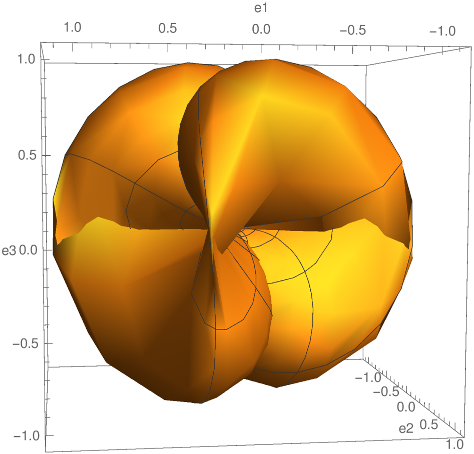
4. The Fokas-Gel'fand formula for immersion (generalized symmetries)

The $\mathfrak{su}(2)$-valued gauges S_k^{FG} associated with ω_k^R have the form

$$S_k^{FG} = (\mathrm{pr}\omega_k^R \Phi_k)\Phi_k^{-1} = D_1 U_{1k} + D_2 U_{2k} = \frac{4}{(t^2+1)(1+|z|^2)^3} \begin{pmatrix} -z^2(1+it)+\bar{z}^2(1-it) & \bar{z}^3(1-it)+z(it+1) \\ -iz^3(t-i)+i\bar{z}(t+i) & z^2(1+it)-\bar{z}^2(1-it) \end{pmatrix}, \qquad k=0,1 \tag{96}$$

where $\det S_k^{FG} \neq 0$.

4. The Fokas-Gel'fand formula for immersion (generalized symmetries)



$$F_k^{FG} = \left(-\frac{x^3 - 6x^2y - x(1+3y^2) + 2y(1+y^2)}{(1+x^2+y^2)^3}, \right.$$
$$\left. \frac{2x^3 + y + 3x^2y - y^3 + x(2-6y^2)}{(1+x^2+y^2)^3}, -\frac{2(x^2-4xy-y^2)}{(1+x^2+y^2)^3} \right)$$

Links between FG and ST immersion formulas

The mapping

$$M_k = S_k^{ST}(S_k^{FG})^{-1}, \qquad k = 0, 1$$

from the FG immersion formulas to the ST immersion formulas are given by

$$M_0 = S_0^{ST}(S_0^{FG})^{-1} = \frac{1}{2(t^2+1)}$$
$$\begin{pmatrix} \frac{-2iz^3(t-i)+\bar{z}[(t+i)^2+|z|^2(t^2+1)]}{z(t-i)} & \frac{z[(t+i)^2+(t^2+1)|z|^2+2(1+it)]}{t-i} \\ -\frac{[z^3(1+t^2)+2\bar{z}(1-it)+z(t-i)^2]}{t+i} & \frac{z(t-i)^2+|z|^2 z(t^2+1)+2i\bar{z}^3(t+i)}{\bar{z}(t+i)} \end{pmatrix},$$

$$M_1 = S_1^{ST}(S_1^{FG})^{-1} = \frac{i}{2|z|^2}$$
$$\begin{pmatrix} \frac{-(1+2|z|^2)}{t^2+1} & \frac{(i+t)^2\bar{z}}{(t-i)^4} \\ \frac{z(t-i)^2}{(t+i)^4} & \frac{1+2|z|^2}{t^2+1} \end{pmatrix}$$
$$\cdot \begin{pmatrix} z^2(it+1)+\bar{z}^2(it-1) & -z(it+1)+\bar{z}^3(it-1) \\ z^3(it+1)+(1-it)\bar{z} & -z^2(1+it)+(1-it)\bar{z}^2 \end{pmatrix},$$

where $\det M_k \neq 0$.

Links between FG and ST immersion formulas

Conversely, there exist mappings from the STIF to the FGIF

$$M_0^{-1} = S_0^{FG}(S_0^{ST})^{-1} = \frac{2}{(1+|z|^2)^3}$$
$$\begin{pmatrix} \frac{(t-i)^2 z+(1+t^2)|z|^2 z+2(it-1)\bar{z}^3}{(i+t)\bar{z}} & \frac{2(1+it)z^2+(i+t)^2\bar{z}^2+(1+t^2)|z|^2\bar{z}^2}{(i-t)z} \\ \frac{(t-i)^2 z^2+(1+t^2)|z|^2 z^2+2(1-it)\bar{z}^2}{(i+t)\bar{z}} & \frac{-2(1+it)z^3+(i+t)^2\bar{z}+(1+t^2)|z|^2\bar{z}}{(t-i)z} \end{pmatrix}, \qquad k=0$$

$$M_1^{-1} = S_1^{FG}(S_1^{ST})^{-1} = \frac{2}{(t^2+1)(1+|z|^2)^3(1+4|z|^2)}$$
$$\begin{pmatrix} -z^2(1+it)+\bar{z}^2(1-it) & \bar{z}^3(1-it)+z(1+it) \\ -z^3(1+it)+\bar{z}(it-1) & z^2(1+it)-\bar{z}^2(1-it) \end{pmatrix}.$$
$$\begin{pmatrix} (1+2|z|^2)(1+it)(i+t) & \frac{-i\bar{z}(i+t)^4}{(t-i)^2} \\ \frac{-iz(t-i)^4}{(i+t)^2} & (1+2|z|^2)(1-it)(-i+t) \end{pmatrix} \qquad k=1$$

where $\det M_k^{-1} \neq 0$.

Applications to ODE's written in the Lax form 1.

Consider an ODE in the independent variable x

$$\Delta[u] \equiv \Delta(x, u, u_x, u_{xx}, ...) = 0, \tag{97}$$

which admits a Lax pair with potential matrices $L(\lambda, [u])$, $M(\lambda, [u])$ taking values in a Lie algebra $\mathfrak{g}$. These matrices satisfy

$$D_x M + [M, L] = 0, \qquad \text{whenever} \qquad \Delta[u] = 0. \tag{98}$$

This Lax representation (98) can be regarded as the compatibility condition of an LSP for a wavefunction Φ taking values in the Lie group G

$$\begin{aligned} D_x\Phi(\lambda, y, [u]) &= L(\lambda, [u])\Phi(\lambda, y, [u]), \\ D_y\Phi(\lambda, y, [u]) &= M(\lambda, [u])\Phi(\lambda, y, [u]). \end{aligned} \tag{99}$$

Here, we have introduced an auxiliary variable y in the LSP for which

$$D_y L = D_y M = 0. \tag{100}$$

ODE's for elliptic equations

Consider a second-order autonomous ODE

$$u_{xx} = \frac{1}{2} f'(u), \qquad f'(u) = \frac{d}{du} f(u) \Leftrightarrow u_x = \epsilon\sqrt{f(u)}, \qquad \epsilon = \pm 1, \tag{101}$$

with solution

$$\int \frac{du}{\epsilon\sqrt{f(u)}} = x - x_0. \tag{102}$$

The ODE (101) admits a Lax pair with potential matrices

$$L = \frac{1}{2} \begin{bmatrix} 0 & \frac{f'(u)}{u+\lambda} - \frac{f(u)-g(\lambda)}{(u+\lambda)^2} \\ 1 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} u_x & -\frac{f(u)-g(\lambda)}{u+\lambda} \\ u+\lambda & -u_x \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{R}). \tag{103}$$

The choice

$$\det M = -g(\lambda) = f(-\lambda) \tag{104}$$

make L and M polynomial in u, whenever $f(u)$ is polynomial in u.

Wavefunctions

The solutions of the wavefunction which satisfy the LSP are denoted by

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \in SL(2, \mathbb{R}) \tag{105}$$

with components

$$\begin{aligned} \Phi_{k1} &= c_1 \Phi_{k+} + c_2 \Phi_{k-}, \qquad k = 1, 2 \\ \Phi_{k2} &= c_3 \Phi_{k+} + c_4 \Phi_{k-}, \qquad c_i \in \mathbb{R}, \qquad i = 1, 2, 3, 4 \end{aligned} \tag{106}$$

and where

$$\begin{aligned} \Phi_{1\pm} &= \frac{\pm\sqrt{g(\lambda)} + u_x}{\sqrt{u+\lambda}} \Psi_{\pm}, \qquad \Phi_{2\pm} = \sqrt{u+\lambda}\,\Psi_{\pm}, \\ \Psi_{\pm} &= \exp\left[\pm\sqrt{g(\lambda)}\left(y + \epsilon \int \frac{du}{2(u+\lambda)\sqrt{f(u)}}\right)\right] \end{aligned} \tag{107}$$

Here the choice of ϵ comes from $u_x = \epsilon\sqrt{f(u)}$. The requirement that $\Phi \in SL(2, \mathbb{R})$ implies

$$c_1 = c_2 = \frac{1}{2}, \qquad c_3 = -c_4 = -\frac{1}{2}\sqrt{g(\lambda)}. \tag{108}$$

Symmetries of ODE's associated with elliptic functions

Consider a vector field in the evolutionary representation

$$v_Q = Q[u]\frac{\partial}{\partial u} \tag{109}$$

which is a generalized symmetry of the ODE (101) iff

$$\begin{aligned} \mathrm{pr}v_Q(u_{xx} - \frac{1}{2}f'(u)) = 0, \qquad \text{whenever} \qquad u_{xx} - \frac{1}{2}f'(u) = 0, \\ \mathrm{pr}v_Q = Q[u]\frac{\partial}{\partial u} + D_J Q\frac{\partial}{\partial u_J} \end{aligned} \tag{110}$$

holds. The determining equation for Q is

$$D_x^2 Q - \frac{1}{2}f''(u)Q = 0, \qquad \text{whenever} \qquad u_{xx} - \frac{1}{2}f'(u) = 0. \tag{111}$$

The following characteristics Q_i's are solutions of the determining equation

$$\begin{array}{cc} Q_1 = u_x & Q_4 = uu_x + xu - \frac{1}{4}x^2u_x \\ Q_2 = u_x\int f(u)^{3/2}du & Q_5 = u^2 - \frac{3}{2}xuu_x - \frac{3}{4}x^2u + \frac{1}{8}x^3u_x \\ Q_3 = xu_x + \gamma u \quad \text{when } f(u) = c_1 + c_2u^l, & l = -2(1+\frac{1}{\gamma}), \qquad \gamma, c_i \in \mathbb{R} \end{array} \tag{112}$$

Case $Q_2 = u_x \int f(u)^{-3/2} du$, $f(u)$-arbitrary function

$$V_{Q_2} = Q_2[u]\frac{\partial}{\partial u}, \qquad Q_2[u] = u_x \int f(u)^{-3/2} du$$

is a symmetry of an elliptic equation (101) but it is not a symmetry of the LSP since the action of $\mathrm{pr}V_{Q_2}$ on the LSP

$$\begin{aligned} \mathrm{pr}v_{Q_2}(D_x\Phi - L\Phi) &= \tfrac{u_x}{2(u+\lambda)^{3/2}\sqrt{f(u)}}A, \\ \mathrm{pr}v_{Q_2}(D_y\Phi - M\Phi) &= \tfrac{u_x}{\sqrt{u+\lambda}\sqrt{f(u)}}A, \end{aligned} \qquad A = \begin{bmatrix} -(\Psi_+ + \Psi_-) & g(\lambda)^{-1/2}(\Psi_+ - \Psi_-) \\ 0 & \Psi_+ + \Psi_- \end{bmatrix} \tag{113}$$

does not vanish for all solutions Φ of the LSP. Thus, there exists an $\mathfrak{sl}(2,\mathbb{R})$-valued immersion function

$$F^{Q_2} = \Phi^{-1}(\mathrm{pr}v_{Q_2}\Phi) \in \mathfrak{sl}(2,\mathbb{R}) \tag{114}$$

with tangent vectors

$$\begin{aligned} D_x F^{Q_2} &= \Phi^{-1}\left[(\mathrm{pr}v_{Q_2}L)\Phi + \mathrm{pr}v_{Q_2}(D_x\Phi - L\Phi)\right], \\ D_y F^{Q_2} &= \Phi^{-1}\left[(\mathrm{pr}v_{Q_2}M)\Phi + \mathrm{pr}v_{Q_2}(D_y\Phi - M\Phi)\right], \end{aligned} \tag{115}$$

Surfaces associated with Jacobi elliptic functions

$$u_x^2 = (1-u^2)(k_1 + k_2 u^2), \qquad k'^2 + k^2 = 1, \qquad 0 \le k, k' \le 1. \tag{116}$$

k_1	k_2	Solution of (116)
1	$-k^2$	$\mathrm{sn}(x,k)$
k'^2	k^2	$\mathrm{cn}(x,k)$
$-k'^2$	1	$\mathrm{dn}(x,k)$

Choosing

$$g(\lambda) = f(-\lambda) = (1-\lambda^2)(k_1 + k_2\lambda^2) \tag{117}$$

the matrices L and M become

$$\begin{aligned} L &= \frac{1}{2}\begin{bmatrix} 0 & -3k_2u^2 + 2\lambda k_2 u + k_1 - k_2 - k_2\lambda^2 \\ 1 & 0 \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \\ M &= \begin{bmatrix} u_x & (u-\lambda)[k_2(u^2+\lambda^2) + k_1 - k_2] \\ u+\lambda & -u_x \end{bmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \end{aligned} \tag{118}$$

Wavefunction and surfaces

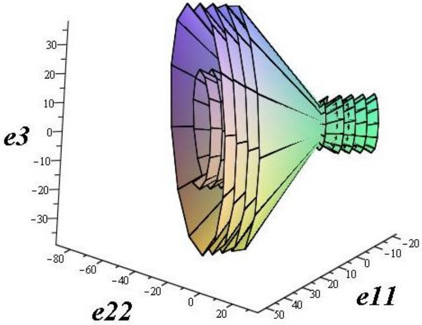
$$\Phi = \begin{bmatrix} \frac{(\sqrt{g(\lambda)}-u_x)\Psi_+ - (\sqrt{g(\lambda)}+u_x)\Psi_-}{2\sqrt{u+\lambda}} & \frac{(\sqrt{g(\lambda)}+u_x\Psi_- - (\sqrt{g(\lambda)}-u_x)\Psi_+)}{2\sqrt{g(\lambda)}\sqrt{u+\lambda}} \\ \frac{\sqrt{u+\lambda}(\Psi_+ + \Psi_-)}{2} & \frac{\sqrt{u+\lambda}(\Psi_- - \Psi_+)}{2\sqrt{g(\lambda)}} \end{bmatrix} \tag{119}$$

where Π is an elliptic integral of the 3rd kind

$$\begin{aligned} \Psi_\pm &= \exp\left[\sqrt{g(\lambda)}(y + \Gamma(u,\lambda))\right], \\ \Gamma(u,\lambda) &= \frac{1}{\lambda\sqrt{k_1}}\Pi(u, \frac{1}{\lambda^2}, \sqrt{\frac{-k_2}{k_1}}) \\ &- \frac{1}{2\sqrt{g}(\lambda)}\tanh^{-1}\left(\frac{(k_2 - k_1 - 2k_2\lambda^2)u^2 + (k_2 - k_1)\lambda^2 + 2k_1}{2\sqrt{g(\lambda)}\sqrt{(1-u^2)(k_1 + k_2 u^2)}}\right) + c_0 \end{aligned} \tag{120}$$

2D-surface $F = \Phi^{-1}(\mathrm{pr} v_Q \Phi)$

Surface $F \in \mathfrak{sl}(2, \mathbb{R})$ for $u = \mathrm{sn}(x, k)$ with $g(\lambda) < 0$, $\lambda = 1.2$ and $x, y \in [-9, 9]$. The axes indicate the components of the immersion function F in the e_i basis of $\mathfrak{sl}(2, \mathbb{R})$. F admits a simple pole



Application to ODE's written in the Lax form 2

Suppose now that the dependent functions $x^k(t)$ depend only on t. The matrices $\mathcal{U}^\alpha$ are functions on the jet space defined by t and $x^k(t)$ and the other independent variable, which here takes the form of a spectral parameter λ. In this case, the ZCC is equivalent to a system of ODE's

$$\Omega[x] = D_\lambda \mathcal{U}^1([x], \lambda) - D_t \mathcal{U}^2([x], \lambda) + [\mathcal{U}^1([x], \lambda), \mathcal{U}^2([x], \lambda)] = 0, \tag{121}$$

where

$$D_t = \frac{\partial}{\partial t} + x_t \frac{\partial}{\partial x} + x_{tt} \frac{\partial}{\partial t} + ..., \qquad D_\lambda = \frac{\partial}{\partial \lambda} \tag{122}$$

The theoretical considerations are illustrated via surfaces associated with the Painlevé P1 equation.

Painlevé P1 surfaces

Here, we present surfaces associated with the Painlevé equation P1

$$\Omega[x] = x_{tt} - 6x^2 - t = 0 \tag{123}$$

The LSP for P1 is given in terms of the potential matrices [Jimbo Miwa 1981]

$$\begin{array}{cc} D_t\Phi = U^1\Phi & D_\lambda\Phi = U^2\Phi \\ \mathcal{U}^1 = \begin{bmatrix} 0 & \lambda + 2x \\ 1 & 0 \end{bmatrix}, & \mathcal{U}^2 = \begin{bmatrix} -x_t & 2\lambda^2 + 2x\lambda + t + 2x^2 \\ 2(\lambda - x) & x_t \end{bmatrix} \in \mathfrak{sl}(2\mathbb{R}) \end{array} \tag{124}$$

which satisfy the ZCC

$$\Omega[x] \equiv D_\lambda\mathcal{U}^1 - D_t\mathcal{U}^2 + [\mathcal{U}^1, \mathcal{U}^2] = (x_{tt} - 6x^2 - t)e_1, \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{125}$$

Painlevé P1 surfaces

Consider the surface F associated with the conformal transformation in the spectral parameter (the ST formula)

$$F = \Phi^{-1}(D_\lambda \Phi) \in \mathfrak{sl}(2, \mathbb{R}) \tag{126}$$

The tangent vectors to the surface F are determined via A^1, A^2

$$D_t F = \Phi^{-1}(D_\lambda \mathcal{U}^1)\Phi, \qquad A_1 = D_\lambda \mathcal{U}^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \tag{127}$$

$$D_\lambda F = \Phi^{-1}(D_\lambda \mathcal{U}^1)\Phi, \qquad A_2 = D_\lambda \mathcal{U}^2 = \begin{pmatrix} 0 & 4\lambda + 2x \\ 2 & 0 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}) \tag{128}$$

The 1st fundamental form associated with the surface F is

$$I(F) = 2dtd\lambda + 4(x + 2\lambda)d\lambda^2 \tag{129}$$

Note that the tangent vector $D_t F$ is an isotropic vector.

Painlevé P1 surfaces

In the moving frame defined by the (nonconstant) wavefunction Φ, the normal to the surface is constant

$$N = \Phi^{-1} e_1 \Phi \in \mathfrak{sl}(2, \mathbb{R}) \tag{130}$$

and so the image of the surface F, written in this moving frame, lies in a plane. The 2nd fundamental form and the Gaussian and mean curvatures for F are

$$\begin{aligned} II(F) &= -dt^2 + 4(x-\lambda)dtdx + 2(4x^2 + 4\lambda x + t - \lambda^2)d\lambda^2 \\ K(F) &= 2(6x^2 + t) = x_{tt} \\ H(F) &= 2(2x + \lambda) \end{aligned} \tag{131}$$

Note that the Gaussian curvature does not depend on λ and the sign of the second derivative of the solution x_{tt} of P1 determines whether the points of F are hyperbolic, elliptic or parabolic.

Painlevé P1 surfaces

The umbilic points of F are determined by

$$H^2 - K = 4(2x+\lambda)^2 - 2x_t t = 0, \qquad x_{tt} = 6x^2 + t \tag{132}$$

which are exactly the curves

$$\lambda = -2x \pm \left(\frac{x_{tt}}{2}\right)^{1/2} \tag{133}$$

There are no umbilic points in the hyperbolic domain where $x_{tt} < 0$. (i.e. $K < 0$)

$$\begin{cases} t = 2(\lambda^2 + x^2 + 4\lambda x) \\ x = -2\lambda \pm \frac{1}{\sqrt{2}}(6\lambda^2 + t)^{1/2} \end{cases} \tag{134}$$

The Laurent series solution of P1 diverge along the curve

$$2(2x+\lambda)^2 - (6x^2 + t) = 0 \tag{135}$$

Concluding remarks

1. We have adapted the Fokas-Gel'fand procedure for constructing soliton surfaces associated with DEs admitting a Lax representation.
2. We have established the connections between three different analytic descriptions for the immersion functions of 2D-surfaces, derived through the links between three types of symmetries: gauge symmetries of the linear spectral problem, conformal transformations in the spectral parameter and generalized symmetries of the integrable system.
3. We have shown that the immersion formulas associated with these symmetries can be linked by gauge transformations.
4. The procedure was applied to the $\mathbb{C}P^{N-1}$ sigma model, and for the elliptic and Painlevé P1 equations.

Future perspectives

1. To use ODE surfaces to approximate PDE surfaces, using group invariant solutions of the integrable PDE. To expand general solutions near group invariant ones through variation of parameters.
2. To use recurrence operators of generalized symmetries of an integrable nonlinear PDE to obtain recurrence relations for surfaces.
3. To investigate how the integrable characteristics, such as Hamiltonian structure and conserved quantities, are manifest in the surfaces.
4. To employ the variational problem of geometric functionals, i.e. the Willmore functional interpreted as an action functional

$$\mathcal{W}(F) = \frac{1}{4}\int_{\Omega} \mathrm{tr}(\mathcal{H}^2)\sqrt{g}\,d\xi\, d\bar{\xi}, \qquad \Omega \subset \mathbb{C}. \tag{136}$$

   to compute the class of equations which are determining equations for the surface (the Euler-Lagrange equations).
5. To develop computer techniques for the visualization of mathematical formulas. A visual image of a surface reflecting the behavior of a solution can be of interest, providing some clues about the properties of this surface, otherwise hidden in some implicit mathematical expressions.

Thank you for your attention.

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Appendix A1: Preliminaries on classical and generalized symmetries

$X \ni x = (x_1, ..., x_p)$, $U \ni u = (u^1, ..., u^q)$ are spaces of independent and dependent variables, respectively.
$J^n = J^n(X \times U)$ is the n-jet space over $X \times U$.
The coordinates of J^n are given by x_α, u^k and

$$u^k_J = \frac{\partial^n u^k}{\partial x_{j_1} ... \partial x_{j_n}} \tag{137}$$

$J = (j_1, ..., j_n)$ is a symmetric multi-index. On J^n we define a system of PDEs

$$\Omega^\mu(x, u^{(n)}) = 0. \qquad \mu = 1, ..., m \tag{138}$$

A vector field v tangent to $J^0 = X \times U$ is denoted by

$$v = \xi^\alpha(x, u)\partial_\alpha + \varphi^k(x, u)\partial_k, \qquad \text{where} \quad \partial_\alpha = \frac{\partial}{\partial x_\alpha}, \quad \partial_k = \frac{\partial}{\partial u^k} \tag{139}$$

$\mathrm{pr}^{(n)} v$ on J^n is a truncated formal series

$$\begin{aligned} &\mathrm{pr}^{(n)} v = \xi^\alpha \partial_\alpha + \varphi^k_J \tfrac{\partial}{\partial u^k_J}. \\ &\varphi^k_J = D_J R^k + \xi^\alpha u^k_{J,\alpha}, \quad R^k = \varphi^k - \xi^\alpha u^k_\alpha, \end{aligned} \tag{140}$$

Appendix A1: Preliminaries on classical and generalized symmetries

The total derivatives are

$$D_\alpha = \partial_\alpha + u^k_{J,\alpha} \frac{\partial}{\partial u^k_J}, \qquad \alpha = 1, \ldots, p \tag{141}$$

and R^k are the so-called characteristics of the vector field v. The representation of v can be written equivalently as

$$v = \xi^\alpha D_\alpha + \omega_R, \qquad \omega_R = R^k \frac{\partial}{\partial u^k} \tag{142}$$

The vector field v is a classical Lie point symmetry of a nondegenerate system of PDEs (138) iff its n-th prolongation of v is such that

$$\mathrm{pr}^{(n)} v \Omega^\mu(x, u^{(n)}) = 0, \qquad \mu = 1, ..., m \tag{143}$$

whenever $\Omega^\mu(x, u^{(n)}) = 0$, $\mu = 1, ..., m$ are satisfied. Every solution of PDEs can be represented by its graph, $u^k = \theta^k(x)$, which is a section of J^0.

Appendix A1: Preliminaries on classical and generalized symmetries

If the graph is preserved by G (equivalently, vectors form $\mathfrak{g}$ are tangent to the graph) then the related solution is said to be G-invariant

$$\Omega(x, \theta^{(n)}) = 0, \qquad \varphi_a^k(x, \theta) - \xi_a^\alpha(x, \theta)\theta^k_{,\alpha} = 0, \qquad a = 1, ..., r \tag{144}$$

A generalized vector field is expressed in terms of the characteristics

$$\omega_R = R^k[u]\frac{\partial}{\partial u^k} \quad \text{where} \quad [u] = (x, u^{(n)}) \in J^n(X \times U). \tag{145}$$

The prolongation of an evolutionary vector field ω_R is given by

$$\mathrm{pr}\omega_R = \omega_R + D_J R^k \frac{\partial}{\partial u^k_J}. \tag{146}$$

A vector field ω_R is a generalized symmetry of a nondegenerated system of PDEs (138) iff

$$\mathrm{pr}\omega_R \Omega^\mu(x, u^{(n)}) = 0, \tag{147}$$

whenever $\Omega(x, u^{(n)}) = 0$ and its differential consequences are satisfied.

Appendix A2: Surfaces associated with $\mathbb{C}P^{N-1}$ models

The surfaces are defined by a contour integral

$$F(\xi,\bar{\xi}) = i\int_\gamma (-[\partial P, P]d\xi + [\bar{\partial}P, P]d\bar{\xi}). \tag{148}$$

The Euler-Lagrange eqs are

$$\partial[\bar{\partial}P, P] + \bar{\partial}[\partial P, P] = 0. \tag{149}$$

The action integral is

$$\int \mathcal{L} d\xi d\bar{\xi} = \mathrm{tr}(\partial P \cdot \bar{\partial}P), \qquad \text{with } P^2 = P, \qquad P^\dagger = P. \tag{150}$$

Eq (149) ensures that (148) is an exact differential. The mapping of $\Omega \subset S^2$ into a set of $\mathfrak{su}(N)$ matrices

$$\Omega \ni (\xi,\bar{\xi}) \mapsto F_k(\xi,\bar{\xi}) \in \mathfrak{su}(N) \simeq \mathbb{R}^{N^2-1}, \qquad 0 \le k \le N-1 \tag{151}$$

is the GWFI of 2D surfaces in $\mathbb{R}^{N^2-1}$.

Appendix A2: Surfaces associated with $\mathbb{C}P^{N-1}$ models

The target spaces of the projectors P_k are 1D vector functions $f_k(\xi,\bar{\xi}) \in \mathbb{C}^N$, constituting an orthogonal basis in $\mathbb{C}^N$

$$P_k = \frac{f_k \otimes f_k^\dagger}{f_k^\dagger f_k}, \qquad P_k P_l = \delta_{kl} P_k \qquad \text{(no summation)}, \qquad \sum_{k=0}^{N-1} P_k = \mathbb{I}_N. \tag{152}$$

All the projectors are obtained form P_0, whose target space is an arbitrary holomorphic vector function $f_0(\xi)$, by the recurrence formulas

$$P_{k-1} = \Pi_-(P_k) = \frac{\bar{\partial} P P \partial P}{\mathrm{tr}(\bar{\partial} P P \partial P)}, \qquad P_{k+1} = \Pi_+(P_k) = \frac{\partial P P \bar{\partial} P}{\mathrm{tr}(\bar{\partial} P P \partial P)}. \tag{153}$$

For the surfaces corresponding to P_k the integration is performed explicitly

$$F_k = -i(P_k + 2\sum_{j=0}^{k-1} P_j) + i c_k \mathbb{I}_N, \qquad c_k = \frac{1}{N}(1+2k). \tag{154}$$

The inverse formulas

$$P_k = F_k^2 - 2i(c_k - 1)F_k - c_k(c_k - 2)\mathbb{I}_N, \qquad 0 \le k \le N-1. \tag{155}$$