Generating sets for $SO(d)$, $SU(d)$

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September 30, 2016

Outline

1. Introduction
2. Problem formulation and possible applications
3. Conditions for generating infinite groups
4. Maximal power N_G
5. Conditions for being dense in a Lie group G
6. Results for $SU(2)$, $SU(3)$, $SO(3)$
7. Summmary

Lie groups, Lie algebras and representation theory

- Lie algebra $\mathfrak{g}$ is simple if it has no nontrivial ideals and semisimple if it is a direct sum of simple ideals.

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k.$$

- G is simple if its Lie algebra is simple. The groups $G = SU(d)$, $G = SO(d)$ are simple except $G = SO(4)$.
- Adjoint representation of G and $\mathfrak{g}$:

$$\mathrm{Ad} : G \to SO(\mathfrak{g}),\ \mathrm{Ad}_g X = gXg^{-1},\ X \in \mathfrak{g},$$
$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{so}(\mathfrak{g}),\ \mathrm{ad}_X(Y) = [X, Y],\ X, Y \in \mathfrak{g}$$

- Adjoint representations of simple Lie algebras and simple Lie groups are irreducible. By Schur lemma:

$$\mathcal{C}(\mathrm{ad}_{\mathfrak{g}}) = \{L \in \mathrm{End}(\mathfrak{g}) : \forall X \in \mathfrak{g}\ [\mathrm{ad}_X, L] = 0\} = \{\lambda I\}$$
$$\mathcal{C}(\mathrm{Ad}_G) = \{L \in \mathrm{End}(\mathfrak{g}) : \forall g \in G\ [\mathrm{Ad}_g, L] = 0\} = \{\lambda I\}$$

- When G compact and connected, then $\exp : \mathfrak{g} \to G$ is onto. For every $g \in G$ there is $X \in \mathfrak{g}$ such that $g = \exp(X)$.

Problem formulation

$\mathcal{S} = \{g_1, \ldots, g_n\} \subset G$, where $G = SU(d)$ or $G = SO(d)$.

$< \mathcal{S} >= G$ if every $g \in G$ can be represented as $g = \prod^{N<\infty} g_{i_j}^{n_{i_j}}$, $g_{i_j} \in \mathcal{S}$

- $< \mathcal{S} >$ is infinite
- $\overline{< \mathcal{S} >}$ is dense in G

Theorems (Kuranishi, da Silva-Leite) [5]

(1) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Then there exist $x, y \in \mathfrak{g}$ s.t. the minimal subalgebra containing x and y is $\mathfrak{g}$.
(2) Let $\mathfrak{t}'$ be a compact real form of a semisimple Lie algebra over $\mathbb{C}$. Then there exist $x, y \in \mathfrak{t}$ s.t. the minimal subalgebra containing x and y is $\mathfrak{t}$.

Theorem 3 (Kuranishi, Field) [5, 4]

Let G be a connected semisimple Lie group. Then there exist the set of pairs $(g, h) \in G$ s.t. the minimal closed subgroup containing g and h is Zariski dense in $G \times G$.

Possible applications in physics

- Lie algebras: control systems

$$\dot{X}(t) = \left(H + \sum_i u_i(t) H_i \right) X(t)$$

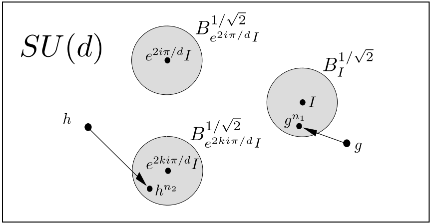
$X(t) \in \mathbb{C}^n$ -state, $H, H_i \in \mathfrak{g}$ - Hamiltonians, $u_i(t) \in \mathbb{R}$ - control amplitudes.

- Lie groups: quantum optics
  - $g \in SU(2)$ - quantum gate acting on a single qubit
  - $g \in \underbrace{SU(2) \otimes \ldots \otimes SU(2)}_{k}$ - gate acting on a system of k qubits separately
  - $g \in SU(2^k)$ - gate acting on a system of k qubits, entangling allowed
  - $g \in SU(d)$ - quantum gate acting on a single qudit (d-state system)

Generating infinite groups

Let $[g,h]_\bullet = ghg^{-1}h^{-1}$ and $\|g\| = \sqrt{\mathrm{tr}(gg^*)}$. Then [2]

$$\|[g,h]_\bullet - I\| \leq \sqrt{2}\|g - I\|\|h - I\|,$$
$$\text{If } [g,[g,h]_\bullet]_\bullet = I \text{ and } \|h - I\| < 2, \text{ then } [g,h]_\bullet = I.$$



Lemma (Sawicki, K.)

Let $g \in B^{1/\sqrt{2}}_{e^{i\theta_m}I}$ and $h \in B^{1/\sqrt{2}}_{e^{i\theta_n}I}$ and assume $[g,h] \notin Z(G)$. The group $< g, h >$ generated by g, h is infinite.

Explicit conditions for $g \in B^{1/\sqrt{2}}_{e^{i\theta_m} I}$

The condition $\|g - e^{i\theta_m} I\| < \frac{1}{\sqrt{2}}$ can be transformed to the form
$\|g - e^{i\theta_m} I\|^2 < \frac{1}{2} \Rightarrow \mathrm{tr}(g - e^{i\theta_m} I)(g^* - e^{-i\theta_m} I) = 2\mathrm{tr} I - e^{-i\theta_m}\mathrm{tr} g - e^{i\theta_m}\mathrm{tr} g^* < \frac{1}{2}$.

- Let $\{e^{i\phi_1}, e^{i\phi_2}, \ldots, e^{i\phi_d}\}$ be the spectrum of $g \in SU(d)$.

$$g \in B_{e^{i\theta_m} I} \Leftrightarrow \sum_{i=1}^{d} \sin^2 \frac{\phi_i - \theta_m}{2} < \frac{1}{8}, \quad \sum_{i=1}^{d} \phi_i = 0 \bmod 2\pi.$$

- Let $\{1, e^{i\phi_1}, e^{-i\phi_1}, \ldots, e^{i\phi_k}, e^{-i\phi_k}\}$, $k = \lfloor \frac{d}{2} \rfloor$ be the spectrum of $g \in SO(d)$.

$$g \in B^{1/\sqrt{2}}_{I} \Leftrightarrow \sum_{i=1}^{k} \sin^2 \frac{\phi_i}{2} < \frac{1}{16},$$

$$g \in B^{1/\sqrt{2}}_{-I} \Leftrightarrow \sum_{i=1}^{k} \sin^2 \frac{\phi_i - \pi}{2} < \frac{1}{16} \ - \text{ only if } d \ - \text{ even}.$$

Maximal power N_G

Maximal power

For groups $G = SU(d)$ and $G = SO(d)$ there is $N_G \in \mathbb{N}$ such that for every $g \in G$, $g^n \in B^{1/\sqrt{2}}_{\alpha_m I}$ for some $e^{i\theta_m} I \in Z(G)$ and $1 \leq n \leq N_G$.

Proof: By Dirichlet approximation theorem and its modification

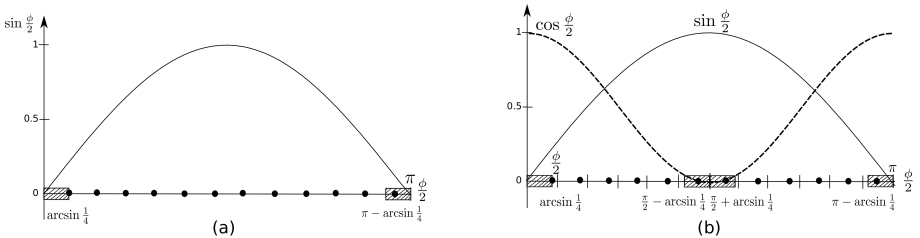
Dirichlet approximation theorem 1

For a given real number a and a positive integer N there exist integers $1 \leq n \leq N$ and p such, that $n\phi$ differs from p by at most $\frac{1}{N+1}$, i.e. $|na - p| \leq \frac{1}{N+1}$.

Dirichlet approximation theorem 2 (modification required [1])

For given real numbers $a_1, \ldots, a_d$ and a positive integer N there exist integer $1 \leq n \leq N$ and integers $p_1, \ldots, p_k$ such that $|na_i - p_i| \leq \frac{1}{(N+1)^{1/d}}$.

Example: $SO(3)$ and $SU(2)$



$$N_{SO(3)} = \lceil \frac{\pi - \arcsin \frac{1}{4}}{\arcsin \frac{1}{4}} \rceil = 12, \quad N_{SU(2)} = \lceil \frac{\pi/2 - \arcsin \frac{1}{4}}{\arcsin \frac{1}{4}} \rceil = 12$$

General case:

$$N_{SO(d)} < \lceil \left(\frac{1}{2}\right)_{d-even} \left(\frac{\pi}{\arcsin \frac{1}{4\sqrt{k}}}\right)^k \rceil, \qquad N_{SU(d)} < \lceil \frac{1}{d} \left(\frac{2\pi}{\beta_d}\right)^{d-1} \rceil$$

where $(d-1)\sin^2 \frac{\beta_d}{2} + \sin^2 \frac{(d-1)\beta_d}{2} = \frac{1}{8}$.

Exceptional spectra of Ad_g

Definition

Assume $g \notin B_{\alpha I}^{1/\sqrt{2}}$ for any $\alpha I \in Z(G)$. A spectral angle ϕ of Ad_g is called an **exceptional angle** iff

1. $\phi \neq 0 \, mod \, \pi$ and there is $n \in \{2, \ldots, N_G\}$ such that $n\phi = 0 \, mod \, \pi$ and $g^n \in B_{\alpha I}^{1/\sqrt{2}}$, or
2. $\phi = (2k+1)\pi$ and there is $n \in \{2, \ldots, N_G\}$ such that $n\phi = 0 \, mod \, 2\pi$ and $g^n \in B_{\alpha I}^{1/\sqrt{2}}$

An exceptional matrix g satisfies $[g^n, h^m]_{\bullet} \in Z(G)$ for some $n, m \in \mathbb{N}$.

- There is $1:1$ correspondence between spectra of Ad_g and g, $g \in SO(d)$
- There is $d-1:1$ correspondence between spectra of Ad_g and g, $g \in SU(d)$

When does infinite $\overline{< \mathcal{S} >}$ is equal to G?

For $\mathfrak{g} = \mathfrak{su}(d), \mathfrak{so}(d)$, $\mathrm{ad}_{\mathfrak{g}}$ is an irreducible representation. By **Schur lemma**

$$\mathcal{C}(\mathrm{ad}_{\mathfrak{g}}) = \{L \in \mathrm{End}(\mathfrak{g}) : \forall x \in \mathfrak{g}\ [\mathrm{ad}_x, L] = 0\} = \{\lambda I\}$$

Let $\mathcal{X} = \{x_1, \ldots, x_k\}$ and $\mathcal{S} = \{g_1, \ldots, g_k\}$, where $g_i = e^{x_i}$. We define

$$\mathcal{C}(\mathrm{ad}_{\mathcal{X}}) = \{L \in \mathrm{End}(\mathfrak{g}) : \forall x_i \in \mathcal{X}\ [\mathrm{ad}_{x_i}, L] = 0\} = \{\lambda I\}$$
$$\mathcal{C}(\mathrm{Ad}_{\mathcal{S}}) = \{L \in \mathrm{End}(\mathfrak{g}) : \forall g_i \in \mathcal{S}\ [\mathrm{Ad}_{g_i}, L] = 0\} = \{\lambda I\}$$

$\mathcal{C}(\mathrm{ad}_{\mathcal{X}}) \subset \mathcal{C}(\mathrm{Ad}_{\mathcal{S}})$. They can differ iff some spectral angles of elements in $\mathrm{Ad}_{\mathcal{S}}$ are $\pm\pi$.

Theorem 1

$\mathcal{X}$ - finite subset of a compact semisimple Lie algebra such that the projection of $\mathcal{X}$ onto every simple component of $\mathfrak{g}$ is nonzero. $< \mathcal{X} >= \mathfrak{g}$ iff $\mathcal{C}(\mathrm{ad}_{\mathfrak{g}}) = \mathcal{C}(\mathrm{ad}_{\mathcal{X}})$.

Theorem 2

$\mathcal{S}$ - finite subset of a compact connected semisimple Lie group G such that $< \mathcal{S} >$ is infinite and the projection of $\mathcal{S}$ onto every simple component of G is also infinite. $< \mathcal{S} >= G$ if and only if $\mathcal{C}(\mathrm{Ad}_G) = \mathcal{C}(\mathrm{Ad}_{\mathcal{S}})$.

Results for $G = SU(2)$

- List of exceptional angles $\mathcal{L}_{SU(2)} = \{0, \pi, \pm\frac{\pi}{2}, \frac{k\pi}{3}, \frac{k\pi}{4}, \frac{k\pi}{5}, \frac{k\pi}{6}\}$
- Let $\mathcal{X} = \{x_1, x_2\}$ where $x_1 = k_x X + k_y Y + k_z Z$ and $x_2 = -k_x X + k_y Y - k_z Z$
- Let $\mathcal{S} = \{g(\phi_1, \vec{k}_1), g(\phi_2, \vec{k}_2)\}$ where $g(\phi_1, \vec{k}_1) = e^{\phi_1 x_1}$ and $g(\phi_2, \vec{k}_2) = e^{\phi_2 x_2}$. By Cayley-Hamilton theorem

$$g(\phi_1, \vec{k}_1) = I \cos\phi_1 + (k_x X + k_y Y + k_z Z)\sin\phi_1,$$
$$g(\phi_2, \vec{k}_2) = I \cos\phi_2 + (-k_x X + k_y Y - k_z Z)\sin\phi_2$$

- $< \mathcal{S} >$ not infinite if $\phi_1, \phi_2 \in \mathcal{L}_{SU(2)}$ and $\vec{k}_1, \vec{k}_2$ are symmetry axes of Platonic polyhedrons
- $\mathcal{C}(\mathrm{Ad}_{\mathcal{S}})$ is larger than $\{\lambda I\}$ if $g(\phi_1, \vec{k}_1), g(\phi_2, \vec{k}_2)$ either commute or anticommute

Universal sets for $SU(d)$, $d > 2$

$\mathcal{S}_{SU(2)} = \{g(\phi, \vec{k}),\ \sigma^t g(\phi, \vec{k})\sigma\}$, where σ - permutation matrix
We embed $g(\phi, \vec{k})$ and $\sigma^t g(\phi, \vec{k})\sigma$ into matrices from $SU(d)$, e.g.

$$g_{12}(\phi) = \begin{pmatrix} I\cos\phi + ik_z\sin\phi & (k_x + ik_y)\sin\phi & 0 \\ (-k_x + ik_y)\sin\phi & I\cos\phi - ik_z\sin\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ etc.}$$

For which $\phi, \vec{k}$ the set $< \mathcal{S} >$ is a generating set?

1. $\phi \notin \{0, \pm\pi, \pm\frac{\pi}{2}\}$ and $\vec{k} \neq \vec{k}_z$ nor $\vec{k} \neq \vec{k}_x$
2. **Any 2-mode orthogonal beamsplitter with $\phi \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$ is universal on 3 and hence $n > 3$ modes.**
3. **Any 2-mode unitary gate whose all entries are nonzero and at least one of them is a complex number is universal on 3 and hence $n > 3$ modes.**

Summary

- Upper bound of N_G
- Definition of exceptional spectra
- Criterion using adjoint representation for deciding if $< g, h >$ is dense in G
- Explicit conditions for 2-element generating sets $\mathcal{S} \subset SU(2), SO(3), SU(3)$ to be generating (universal) sets
- *All* the pairs of generators of finite $SU(2)$ subgroups listed
- Non-generating sets in $SU(2)$ and $S0(2)$ become generating in almost all cases after embedding into $SU(d)$ and $SO(d)$ respectively, $d \geq 3$

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