On classification of Lie pencils

50th seminar "Sophus Lie", Bedlewo, 25 September - 1 October 2016

Andriy Panasyuk

Faculty of Mathematics and Computer Science
University of Warmia and Mazury
Olsztyn, Poland

Based on:

"Compatible Lie brackets: Towards a Classification"
Journal of Lie Theory, Volume 24 (2014) 561-623

Main definition

Definition

A *bi-Lie structure* is a triple $(\mathfrak{g}, [,], [,]')$, where $\mathfrak{g}$ is a vector space and $[,], [,]'$ are two Lie brackets on $\mathfrak{g}$ which are *compatible*, i.e. so that $[,] + [,]'$ is a Lie bracket.

Example

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, $A \in \mathfrak{g}$ be a fixed matrix. Put

$$[x,_A y] = xAy - yAx.$$

Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure, ($[,]$ the standard commutator).

Main motivating example

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K})$, $A \in \mathrm{Symm}(n, \mathbb{K})$, a fixed symmetric matrix. Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure.

Main definition

Definition

A *bi-Lie structure* is a triple $(\mathfrak{g}, [,], [,]')$, where $\mathfrak{g}$ is a vector space and $[,], [,]'$ are two Lie brackets on $\mathfrak{g}$ which are *compatible*, i.e. so that $[,] + [,]'$ is a Lie bracket.

Example

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, $A \in \mathfrak{g}$ be a fixed matrix. Put

$$[x,_A y] = xAy - yAx.$$

Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure, ($[,]$ the standard commutator).

Main motivating example

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K})$, $A \in \mathrm{Symm}(n, \mathbb{K})$, a fixed symmetric matrix. Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure.

Main definition

Definition

A *bi-Lie structure* is a triple $(\mathfrak{g}, [,], [,]')$, where $\mathfrak{g}$ is a vector space and $[,], [,]'$ are two Lie brackets on $\mathfrak{g}$ which are *compatible*, i.e. so that $[,] + [,]'$ is a Lie bracket.

Example

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, $A \in \mathfrak{g}$ be a fixed matrix. Put

$$[x,_A y] = xAy - yAx.$$

Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure, ($[,]$ the standard commutator).

Main motivating example

Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K})$, $A \in \mathrm{Symm}(n, \mathbb{K})$, a fixed symmetric matrix. Then $(\mathfrak{g}, [,], [,_A])$ is a bi-Lie structure.

Motivation I: bihamiltonian structures

Definition

A *bihamiltonian* structure on a manifold M is a pair $\eta_1, \eta_2 \in \Gamma(\bigwedge^2 TM)$ such that $\eta_1, \eta_2, \eta_1 + \eta_2$ are Poisson.

Hierarchy of mechanisms (by complexity of structures):

- constant+constant (rather not interesting)
- constant+linear (proved to be powerful, eg. "argument translation")
- linear+linear (topic of present talk)
- linear+quadratic (eg. argument translation of quadratic bracket towards "vanishing direction")
- etc.

Motivation I: bihamiltonian structures

Definition

A *bihamiltonian* structure on a manifold M is a pair $\eta_1, \eta_2 \in \Gamma(\bigwedge^2 TM)$ such that $\eta_1, \eta_2, \eta_1 + \eta_2$ are Poisson.

Hierarchy of mechanisms (by complexity of structures):

- constant+constant (rather not interesting)
- constant+linear (proved to be powerful, eg. "argument translation")
- linear+linear (topic of present talk)
- linear+quadratic (eg. argument translation of quadratic bracket towards "vanishing direction")
- etc.

Motivation I: bihamiltonian structures

Semisimple case

Applications of the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure:

- Manakov top (n-dimensional free rigid body), here A is diagonal, the "inertia tensor" of the body (Bolsinov 1992)
- Klebsh–Perelomov case (Bolsinov 1992)

Another bi-Lie structure on $\mathfrak{so}(n, \mathbb{R}) \times \mathfrak{so}(n, \mathbb{R})$

- Generalized Steklov–Lyapunov systems (Bolsinov–Fedorov 1992)

Nonsemisimple case

Works of Golubchik, Odesskii, Sokolov $\sim$ 2004–2006

- Matrix integrable ODE's

Motivation I: bihamiltonian structures

Semisimple case

Applications of the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure:

- Manakov top (n-dimensional free rigid body), here A is diagonal, the "inertia tensor" of the body (Bolsinov 1992)
- Klebsh–Perelomov case (Bolsinov 1992)

Another bi-Lie structure on $\mathfrak{so}(n, \mathbb{R}) \times \mathfrak{so}(n, \mathbb{R})$

- Generalized Steklov–Lyapunov systems (Bolsinov–Fedorov 1992)

Nonsemisimple case

Works of Golubchik, Odesskii, Sokolov $\sim$ 2004–2006

- Matrix integrable ODE's

Motivation II: classical R-matrix formalism

Classical R-matrix

Let $\mathfrak{g}$ be a Lie algebra and $R : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$ {}_R[x, y] := [Rx, y] + [x, Ry]. $$

We say that R is a classical R-matrix if ${}_R[,]$ is a Lie bracket.

Standard classical R-matrix

Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ are subalgebras. Then $R := P_+ - P_-$ is a classical R-matrix called *standard*.

Basic example of the standard classical R-matrix

Let $\mathfrak{g}$ be a Lie algebra, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Then $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n$ is the space of homogeneous Laurent polynomials of degree n and $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras in $\tilde{\mathfrak{g}}$.

Motivation II: classical R-matrix formalism

Classical R-matrix

Let $\mathfrak{g}$ be a Lie algebra and $R : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$_R[x, y] := [Rx, y] + [x, Ry].$$

We say that R is a classical R-matrix if $_R[,]$ is a Lie bracket.

Standard classical R-matrix

Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ are subalgebras. Then $R := P_+ - P_-$ is a classical R-matrix called *standard*.

Basic example of the standard classical R-matrix

Let $\mathfrak{g}$ be a Lie algebra, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Then $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n$ is the space of homogeneous Laurent polynomials of degree n and $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras in $\tilde{\mathfrak{g}}$.

Motivation II: classical R-matrix formalism

Classical R-matrix

Let $\mathfrak{g}$ be a Lie algebra and $R : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$_R[x, y] := [Rx, y] + [x, Ry].$$

We say that R is a classical R-matrix if $_R[,]$ is a Lie bracket.

Standard classical R-matrix

Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ are subalgebras. Then $R := P_+ - P_-$ is a classical R-matrix called *standard*.

Basic example of the standard classical R-matrix

Let $\mathfrak{g}$ be a Lie algebra, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Then $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n$ is the space of homogeneous Laurent polynomials of degree n and $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras in $\tilde{\mathfrak{g}}$.

Motivation II: classical R-matrix formalism

Quasigraded Lie algebras

A Lie algebra $(\tilde{\mathfrak{g}}, [,])$ with a decomposition $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is *quasigraded of degree 1* if $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}$

Quasigraded Lie algebras → standard classical R-matrix

One checks that $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras.

Bi-Lie structures → quasigraded Lie algebras

Let $(\mathfrak{g}, [,]_0, [,]_1)$ be a bi-Lie structure, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Put $[,] = [,]_0 + \lambda[,]_1$ and extend this bracket to $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{g}}$ is quasigraded of degree 1.

Motivation II: classical R-matrix formalism

Quasigraded Lie algebras

A Lie algebra $(\tilde{\mathfrak{g}}, [,])$ with a decomposition $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is *quasigraded of degree 1* if $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}$

Quasigraded Lie algebras → standard classical R-matrix

One checks that $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras.

Bi-Lie structures → quasigraded Lie algebras

Let $(\mathfrak{g}, [,]_0, [,]_1)$ be a bi-Lie structure, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Put $[,] = [,]_0 + \lambda[,]_1$ and extend this bracket to $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{g}}$ is quasigraded of degree 1.

Motivation II: classical R-matrix formalism

Quasigraded Lie algebras

A Lie algebra $(\tilde{\mathfrak{g}}, [,])$ with a decomposition $\tilde{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is *quasigraded of degree 1* if $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}$

Quasigraded Lie algebras $\to$ standard classical R-matrix

One checks that $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n, \mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are subalgebras.

Bi-Lie structures $\to$ quasigraded Lie algebras

Let $(\mathfrak{g}, [,]_0, [,]_1)$ be a bi-Lie structure, $\tilde{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$. Put $[,] = [,]_0 + \lambda[,]_1$ and extend this bracket to $\tilde{\mathfrak{g}}$. Then $\tilde{\mathfrak{g}}$ is quasigraded of degree 1.

Motivation II: classical R-matrix formalism

Applications

- Landau-Livshits PDE (the $\mathfrak{so}(n, \mathbb{R})$ bi-Lie structure, $n = 3$, Holod 1987)
- Other finite- and infinite-dimensional systems (Skrypnyk, Golubchik–Sokolov, Yanovski)

Known classification results: the Kantor–Persits theorem

Useful notation

Let $\mathfrak{g}$ be a Lie algebra and $N : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$[x, y]_N := [Nx, y] + [x, Ny] - N[x, y].$$

Definition

Let $\{[,]^v\}_{v \in V}$ be a n-dimensional vector space of Lie structures on a vector space $\mathfrak{g}$. It is called *irreducible* if the Lie algebras $(\mathfrak{g}, [,]^v)$ do not have common nontrivial ideals and *closed* if

$$\forall x \in \mathfrak{g}\ \forall v, w \in V\ \exists u \in V : [,]^v_{\mathrm{ad}^w x} := [,]^u, \mathrm{ad}^w x(y) = [x, y]^w.$$

Known classification results: the Kantor–Persits theorem

Useful notation

Let $\mathfrak{g}$ be a Lie algebra and $N : \mathfrak{g} \to \mathfrak{g}$ a linear operator. Put

$$[x, y]_N := [Nx, y] + [x, Ny] - N[x, y].$$

Definition

Let $\{[,]^v\}_{v \in V}$ be a n-dimensional vector space of Lie structures on a vector space $\mathfrak{g}$. It is called *irreducible* if the Lie algebras $(\mathfrak{g}, [,]^v)$ do not have common nontrivial ideals and *closed* if

$$\forall x \in \mathfrak{g}\ \forall v, w \in V\ \exists u \in V : [,]^v_{\mathrm{ad}^w x} := [,]^u, \mathrm{ad}^w x(y) = [x, y]^w.$$

Known classification results: the Kantor–Persits theorem

Kantor–Persits 1988 (announced only)

The list of irreducible closed vector spaces of Lie structures:

- $\mathfrak{g} = \mathfrak{so}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathrm{Symm}(n,\mathbb{K})}$
- $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{K}), \{[,_A]\}_{A \in \mathfrak{m}(n,\mathbb{K})}$
- several *nonsemisimple* cases

here

$$[X,_A Y] := XAY - YAX,$$

$\mathfrak{sp}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ + JX^T = 0\}$ the symplectic Lie algebra, $\mathfrak{m}(n, \mathbb{K}) := \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid XJ - JX^T = 0\}$ its orthogonal complement in $\mathfrak{gl}(2n, \mathbb{K})$ w.r.t. "trace form"

Known classification results: the Odesskii–Sokolov theorem

Odesskii–Sokolov 2006

Classification of "bi-associative structures" $(\cdot, \circ)$ on $\mathfrak{gl}(n, \mathbb{K}) \Longrightarrow$ Examples of bi-Lie structures on $\mathfrak{gl}(n, \mathbb{K})$ (which do not restrict to $\mathfrak{sl}(n, \mathbb{K})$)

Semisimple bi-Lie structures and their examples

Definition

Say that a bi-Lie structure $\mathcal{B} := (\mathfrak{g}, [,], [,]')$ is *semisimple* if $(\mathfrak{g}, [,])$ is semisimple.

Known examples of semisimple bi-Lie structures

KP1 $(\mathfrak{so}(n, \mathbb{C}), [,], [,_A])$ (Kantor–Persits 1988)

KP2 $(\mathfrak{sp}(n, \mathbb{C}), [,], [,_A])$ (Kantor–Persits 1988)

GS1 Let $(\mathfrak{g}, [,])$ be semisimple. There exists a bi-Lie structure related to any $\mathbb{Z}_n$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ on $(\mathfrak{g}, [,])$ and to decomposition of the subalgebra $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$ to two subalgebras (Golubchik–Sokolov 2002)

P Let $(\mathfrak{g}, [,])$ be semisimple. There exists a bi-Lie structure related to any parabolic subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ (P 2006)

GS2 Examples on $\mathfrak{sl}(3, \mathbb{C}), \mathfrak{so}(4, \mathbb{C})$ related to $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings (Golubchik–Sokolov 2002)

Semisimple bi-Lie structures and operators

Obvious or Easy:

Let $(\mathfrak{g}, [,])$ *be a Lie algebra,* $[,]'$ *a bilinear bracket.*

- $[,]'$ *"compatible" with* $[,] \iff [,]'$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular, if* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str., then* $[,]' = [,]_W = [W\cdot, \cdot] + [\cdot, W\cdot] - W[\cdot, \cdot]$ *for some* $W : \mathfrak{g} \to \mathfrak{g}$
- *(Magri–Kosmann-Schwarzbach)* $[,]_N$ *is a Lie bracket for some* $N : \mathfrak{g} \to \mathfrak{g} \iff T_N(\cdot, \cdot) := [N\cdot, N\cdot] - N[\cdot, \cdot]_N$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular,* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str.* $\iff [,]' = [,]_W$ *and* $T_W(\cdot, \cdot) = [\cdot, \cdot]_P$*, where* $P : \mathfrak{g} \to \mathfrak{g}$ *is another linear operator. Moreover, the operators* W, P *are defined up to adding of inner differentiations* $\operatorname{ad} x$*.*

$$T_N(X, Y) := [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y])$$
$$= [PX, Y] + [X, PY] - P[X, Y] \quad \text{(MI)}$$

Semisimple bi-Lie structures and operators

Obvious or Easy:

Let $(\mathfrak{g}, [,])$ *be a Lie algebra,* $[,]'$ *a bilinear bracket.*

- $[,]'$ *"compatible" with* $[,] \Longleftrightarrow [,]'$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular, if* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str., then* $[,]' = [,]_W = [W\cdot, \cdot] + [\cdot, W\cdot] - W[\cdot, \cdot]$ *for some* $W : \mathfrak{g} \to \mathfrak{g}$
- *(Magri–Kosmann-Schwarzbach)* $[,]_N$ *is a Lie bracket for some* $N : \mathfrak{g} \to \mathfrak{g} \Longleftrightarrow T_N(\cdot, \cdot) := [N\cdot, N\cdot] - N[\cdot, \cdot]_N$ *is a 2-cocycle on* $(\mathfrak{g}, [,])$
- *In particular,* $(\mathfrak{g}, [,], [,]')$ *is a semisimple bi-Lie str.* $\Longleftrightarrow [,]' = [,]_W$ *and* $T_W(\cdot, \cdot) = [\cdot, \cdot]_P$*, where* $P : \mathfrak{g} \to \mathfrak{g}$ *is another linear operator. Moreover, the operators* W, P *are defined up to adding of inner differentiations* $\operatorname{ad} x$.

$$T_N(X, Y) := [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y])$$
$$= [PX, Y] + [X, PY] - P[X, Y] \quad \text{(MI)}$$

Semisimple bi-Lie structures and operators

Obvious or Easy:

Let $(\mathfrak{g}, [\,,\,])$ *be a Lie algebra,* $[\,,\,]'$ *a bilinear bracket.*

- $[\,,\,]'$ *"compatible" with* $[\,,\,]$ $\Longleftrightarrow$ $[\,,\,]'$ *is a 2-cocycle on* $(\mathfrak{g}, [\,,\,])$
- *In particular, if* $(\mathfrak{g}, [\,,\,], [\,,\,]')$ *is a semisimple bi-Lie str., then* $[\,,\,]' = [\,,\,]_W = [W\cdot, \cdot] + [\cdot, W\cdot] - W[\cdot, \cdot]$ *for some* $W : \mathfrak{g} \to \mathfrak{g}$
- *(Magri–Kosmann-Schwarzbach)* $[\,,\,]_N$ *is a Lie bracket for some* $N : \mathfrak{g} \to \mathfrak{g} \Longleftrightarrow T_N(\cdot, \cdot) := [N\cdot, N\cdot] - N[\cdot, \cdot]_N$ *is a 2-cocycle on* $(\mathfrak{g}, [\,,\,])$
- *In particular,* $(\mathfrak{g}, [\,,\,], [\,,\,]')$ *is a semisimple bi-Lie str.* $\Longleftrightarrow$ $[\,,\,]' = [\,,\,]_W$ *and* $T_W(\cdot, \cdot) = [\cdot, \cdot]_P$*, where* $P : \mathfrak{g} \to \mathfrak{g}$ *is another linear operator. Moreover, the operators* W, P *are defined up to adding of inner differentiations* $\operatorname{ad} x$*.*

$$T_N(X, Y) := [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y])$$
$$= [PX, Y] + [X, PY] - P[X, Y] \quad \text{(MI)}$$

Semisimple bi-Lie structures and operators

Obvious or Easy:

Let $(\mathfrak{g},[,])$ *be a Lie algebra,* $[,]'$ *a bilinear bracket.*

- $[,]'$ *"compatible" with* $[,]$ $\Longleftrightarrow$ $[,]'$ *is a 2-cocycle on* $(\mathfrak{g},[,])$
- *In particular, if* $(\mathfrak{g},[,],[,]')$ *is a semisimple bi-Lie str., then* $[,]'=[,]_W=[W\cdot,\cdot]+[\cdot,W\cdot]-W[\cdot,\cdot]$ *for some* $W:\mathfrak{g}\to\mathfrak{g}$
- *(Magri–Kosmann-Schwarzbach)* $[,]_N$ *is a Lie bracket for some* $N:\mathfrak{g}\to\mathfrak{g}\Longleftrightarrow T_N(\cdot,\cdot):=[N\cdot,N\cdot]-N[\cdot,\cdot]_N$ *is a 2-cocycle on* $(\mathfrak{g},[,])$
- *In particular,* $(\mathfrak{g},[,],[,]')$ *is a semisimple bi-Lie str.* $\Longleftrightarrow$ $[,]'=[,]_W$ *and* $T_W(\cdot,\cdot)=[\cdot,\cdot]_P$*, where* $P:\mathfrak{g}\to\mathfrak{g}$ *is another linear operator. Moreover, the operators* W,P *are defined up to adding of inner differentiations* $\mathrm{ad}\,x$.

$$T_N(X,Y):=[NX,NY]-N([NX,Y]+[X,NY]-N[X,Y])$$
$$=[PX,Y]+[X,PY]-P[X,Y] \quad \text{(MI)}$$

Semisimple bi-Lie structures: examples of leading operators

Definition

Given a semisimple bi-Lie structure $\mathcal{B}$ call W such that $[,]' = [,]_W$ a *leading operator* for $\mathcal{B}$ and P a *primitive* for W. They satisfy *the main identity (MI)*

$$T_W(\cdot,\cdot) = [\cdot,\cdot]_P$$

Example

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$. Put $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ and $P|_{\mathfrak{g}_i} = \frac{1}{2}i(n-i)\mathrm{Id}_{\mathfrak{g}_i}$. One checks MI directly.

Example

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (sum of subalgebras). Put $W|_{\mathfrak{g}_i} = \omega_i \mathrm{Id}_{\mathfrak{g}_i}, i = 1, 2$, where $\omega_{1,2}$ are any scalars. Then $T_W = 0$ (so put $P = 0$ in the MI). Important example: $\mathfrak{g}$ simple, $\mathfrak{g}_1$ a parabolic subalgebra and $\mathfrak{g}_2$ its "complement".

Semisimple bi-Lie structures: examples of leading operators

Definition

Given a semisimple bi-Lie structure $\mathcal{B}$ call W such that $[,]' = [,]_W$ a *leading operator* for $\mathcal{B}$ and P a *primitive* for W. They satisfy *the main identity (MI)*

$$T_W(\cdot,\cdot) = [\cdot,\cdot]_P$$

Example

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ *be a* $\mathbb{Z}_n$*-grading on* $\mathfrak{g}$*. Put* $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ *and* $P|_{\mathfrak{g}_i} = \frac{1}{2}i(n-i)\mathrm{Id}_{\mathfrak{g}_i}$*. One checks MI directly.*

Example

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (sum of subalgebras). Put $W|_{\mathfrak{g}_i} = \omega_i\mathrm{Id}_{\mathfrak{g}_i}, i = 1, 2$, where $\omega_{1,2}$ are any scalars. Then $T_W = 0$ (so put $P = 0$ in the MI). Important example: $\mathfrak{g}$ simple, $\mathfrak{g}_1$ a parabolic subalgebra and $\mathfrak{g}_2$ its “complement”.

Semisimple bi-Lie structures: examples of leading operators

Definition

Given a semisimple bi-Lie structure $\mathcal{B}$ call W such that $[,]' = [,]_W$ a *leading operator* for $\mathcal{B}$ and P a *primitive* for W. They satisfy *the main identity (MI)*

$$T_W(\cdot,\cdot) = [\cdot,\cdot]_P$$

Example

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ *be a* $\mathbb{Z}_n$*-grading on* $\mathfrak{g}$*. Put* $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ *and* $P|_{\mathfrak{g}_i} = \frac{1}{2}i(n-i)\mathrm{Id}_{\mathfrak{g}_i}$*. One checks MI directly.*

Example

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (sum of subalgebras). Put $W|_{\mathfrak{g}_i} = \omega_i \mathrm{Id}_{\mathfrak{g}_i}, i = 1, 2$, where $\omega_{1,2}$ are any scalars. Then $T_W = 0$ (so put $P = 0$ in the MI). Important example: $\mathfrak{g}$ simple, $\mathfrak{g}_1$ a parabolic subalgebra and $\mathfrak{g}_2$ its “complement”.

Principal leading operator

Definition

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a direct decomposition $\mathrm{End}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \oplus C$, where $C = (\mathrm{ad}\,\mathfrak{g})^{\perp}$ is the orthogonal complement to $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ w.r.t. the trace form. An operator $W \in \mathrm{End}(\mathfrak{g})$ is called *principal* if $W \in C$.

Theorem

1. *There exists a unique principal operator W with the property $[,]' = [,]_W$. Call it the* principal *(leading) operator of a bi-Lie structure $(\mathfrak{g}, [,], [,]')$.*
2. *If W is the principal operator, there exists a unique operator P primitive for W which is* symmetric *w.r.t. the trace form on $\mathrm{End}(\mathfrak{g})$.*

Example

For $\mathfrak{so}(n, \mathbb{K})$ bi-Lie structure we have $W = (1/2)(L_A + R_A)$ (operators of left and right multiplication by A).

Principal leading operator

Definition

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a direct decomposition $\mathrm{End}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \oplus C$, where $C = (\mathrm{ad}\,\mathfrak{g})^{\perp}$ is the orthogonal complement to $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ w.r.t. the trace form. An operator $W \in \mathrm{End}(\mathfrak{g})$ is called *principal* if $W \in C$.

Theorem

1. *There exists a unique principal operator W with the property $[\,,]' = [\,,]_W$. Call it the* principal *(leading) operator of a bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$.*
2. *If W is the principal operator, there exists a unique operator P primitive for W which is* symmetric *w.r.t. the trace form on $\mathrm{End}(\mathfrak{g})$.*

Example

For $\mathfrak{so}(n, \mathbb{K})$ bi-Lie structure we have $W = (1/2)(L_A + R_A)$ (operators of left and right multiplication by A).

Principal leading operator

Definition

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a direct decomposition $\mathrm{End}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g} \oplus C$, where $C = (\mathrm{ad}\,\mathfrak{g})^{\perp}$ is the orthogonal complement to $\mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$ w.r.t. the trace form. An operator $W \in \mathrm{End}(\mathfrak{g})$ is called *principal* if $W \in C$.

Theorem

1. *There exists a unique principal operator W with the property $[,]' = [,]_W$. Call it the* principal *(leading) operator of a bi-Lie structure $(\mathfrak{g}, [,], [,]')$.*
2. *If W is the principal operator, there exists a unique operator P primitive for W which is* symmetric *w.r.t. the trace form on $\mathrm{End}(\mathfrak{g})$.*

Example

For $\mathfrak{so}(n, \mathbb{K})$ bi-Lie structure we have $W = (1/2)(L_A + R_A)$ (operators of left and right multiplication by A).

Significance of the principal leading operator

Definition

We say that bi-Lie structures $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ are *strongly isomorphic* (*isomorphic*) if there exists an automorphism of the Lie algebra $(\mathfrak{g}, [,])$ sending the bracket $[,]'$ to $[,]''$ (to a linear combination $\alpha_1[,] + \alpha_2[,]''$).

Theorem

Let $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ be two semisimple bi-Lie structures and let W', W'' be the corresponding principal operators. Then the bi-Lie structures are strongly isomorphic if and only if there exists an automorphism ϕ of the Lie algebra $(\mathfrak{g}, [,])$ with the property $\phi \circ W' = W'' \circ \phi$.

In particular, classification of semisimple bi-Lie structures up to isomorphism $\iff$ classification of principal operators satisfyting MI up to action of automorphisms, rescaling, and adding scalar operators

Significance of the principal leading operator

Definition

We say that bi-Lie structures $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ are *strongly isomorphic* (*isomorphic*) if there exists an automorphism of the Lie algebra $(\mathfrak{g}, [,])$ sending the bracket $[,]'$ to $[,]''$ (to a linear combination $\alpha_1[,] + \alpha_2[,]''$).

Theorem

Let $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ be two semisimple bi-Lie structures and let W', W'' be the corresponding principal operators. Then the bi-Lie structures are strongly isomorphic if and only if there exists an automorphism ϕ of the Lie algebra $(\mathfrak{g}, [,])$ with the property $\phi \circ W' = W'' \circ \phi$.

In particular, classification of semisimple bi-Lie structures up to isomorphism $\Longleftrightarrow$ classification of principal operators satisfyting MI up to action of automorphisms, rescaling, and adding scalar operators

Significance of the principal leading operator

Definition

We say that bi-Lie structures $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ are *strongly isomorphic* (*isomorphic*) if there exists an automorphism of the Lie algebra $(\mathfrak{g}, [,])$ sending the bracket $[,]'$ to $[,]''$ (to a linear combination $\alpha_1[,] + \alpha_2[,]''$).

Theorem

Let $(\mathfrak{g}, [,], [,]')$ and $(\mathfrak{g}, [,], [,]'')$ be two semisimple bi-Lie structures and let W', W'' be the corresponding principal operators. Then the bi-Lie structures are strongly isomorphic if and only if there exists an automorphism ϕ of the Lie algebra $(\mathfrak{g}, [,])$ with the property $\phi \circ W' = W'' \circ \phi$.

In particular, classification of semisimple bi-Lie structures up to isomorphism $\Longleftrightarrow$ classification of principal operators satisfyting MI up to action of automorphisms, rescaling, and adding scalar operators

The pencil of Lie algebras and the times

Switch to $\mathbb{K} = \mathbb{C}$

Bi-Lie structure $(\mathfrak{g}, [,], [,]') \Longrightarrow$ Pencil of Lie brackets $(\mathfrak{g}, [,]^t), [,]^t := [,]' - t[,], t \in \mathbb{C}$

Theorem

Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure, W its principal operator, P its symmetric primitive and let $B(,)$ be the Killing form of $(\mathfrak{g}, [,])$. Then the Killing form B^t of the Lie algebra $(\mathfrak{g}, [,]^t)$ is given by the formula

$$B^t(x, y) = B((W - tI)x, (W - tI)y) - 2B(Px, y), \; x, y \in \mathfrak{g},$$

In particular, $\ker B^t \neq \{0\} \Longleftrightarrow \det(W^*W - 2P - t(W + W^*) + t^2 I) = 0.$

Definition

The elements of the finite set $T := \{t \in \mathbb{C} \mid \ker B^t \neq \{0\}\}$ are called the *times* of the bi-Lie structure.

The pencil of Lie algebras and the times

Switch to $\mathbb{K} = \mathbb{C}$

Bi-Lie structure $(\mathfrak{g}, [,], [,]') \Longrightarrow$ Pencil of Lie brackets $(\mathfrak{g}, [,]^t), [,]^t := [,]' - t[,], t \in \mathbb{C}$

Theorem

Let $(\mathfrak{g}, [,], [,]')$ *be a semisimple bi-Lie structure,* W *its principal operator,* P *its symmetric primitive and let* $B(,)$ *be the Killing form of* $(\mathfrak{g}, [,])$. *Then the Killing form* B^t *of the Lie algebra* $(\mathfrak{g}, [,]^t)$ *is given by the formula*

$$B^t(x, y) = B((W - tI)x, (W - tI)y) - 2B(Px, y), \; x, y \in \mathfrak{g},$$

In particular, $\ker B^t \neq \{0\} \Longleftrightarrow \det(W^*W - 2P - t(W + W^*) + t^2 I) = 0.$

Definition

The elements of the finite set $T := \{t \in \mathbb{C} \mid \ker B^t \neq \{0\}\}$ are called the *times* of the bi-Lie structure.

The central subalgebra

In particular, if $t \in T$, the center $\mathfrak{z}^t$ of the Lie algebra $(\mathfrak{g}, [,]^t)$ can be nontrivial.

Theorem

1. *The subset* $\mathfrak{z}^t$ *is a subalgebra in* $(\mathfrak{g}, [,])$ *for any* $t \in T$*;*
2. $\mathfrak{z}^{t_1} \cap \mathfrak{z}^{t_2} = \{0\}$ *and* $[\mathfrak{z}^{t_1}, \mathfrak{z}^{t_2}] = 0$ *if* $t_1 \neq t_2$*;*
3. *in particular, the set* $\mathfrak{z} := \sum_{t \in T} \mathfrak{z}^t$ *is a subalgebra in* $(\mathfrak{g}, [,])$ *which is a direct sum of its ideals* $\mathfrak{z}^{t_i}$*. Call* $\mathfrak{z}$ *the* central subalgebra *of* $(\mathfrak{g}, [,], [,]')$*. Moreover,* $\mathfrak{z} \subset \ker P$*.*

Examples: (1) $(\mathfrak{so}(6, \mathbb{C}), [,], [,_A])$; (2) $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$

$$(1) A = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}, \mathfrak{z} = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; (2) \mathfrak{z} = \mathfrak{g}_0$$

The central subalgebra

In particular, if $t \in T$, the center $\mathfrak{z}^t$ of the Lie algebra $(\mathfrak{g}, [,]^t)$ can be nontrivial.

Theorem

1. *The subset* $\mathfrak{z}^t$ *is a subalgebra in* $(\mathfrak{g}, [,])$ *for any* $t \in T$;
2. $\mathfrak{z}^{t_1} \cap \mathfrak{z}^{t_2} = \{0\}$ *and* $[\mathfrak{z}^{t_1}, \mathfrak{z}^{t_2}] = 0$ *if* $t_1 \neq t_2$;
3. *in particular, the set* $\mathfrak{z} := \sum_{t \in T} \mathfrak{z}^t$ *is a subalgebra in* $(\mathfrak{g}, [,])$ *which is a direct sum of its ideals* $\mathfrak{z}^{t_i}$. *Call* $\mathfrak{z}$ *the* central subalgebra *of* $(\mathfrak{g}, [,], [,]')$. *Moreover,* $\mathfrak{z} \subset \ker P$.

Examples: (1) $(\mathfrak{so}(6, \mathbb{C}), [,], [,_A])$; (2) $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$

$$(1) A = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & c \end{bmatrix}, \mathfrak{z} = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; (2) \mathfrak{z} = \mathfrak{g}_0$$

Gradings and Main assumption

Definition

Let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a *grading* of a Lie algebra $(\mathfrak{g}, [,])$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i, j \in \Gamma$, Γ an abelian group. We say that a linear operator $W : \mathfrak{g} \to \mathfrak{g}$ *preserves* the grading if $W\mathfrak{g}_i \subset \mathfrak{g}_i$ for any $i \in \Gamma$.

Theorem

Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure and let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a grading. Then, if the principal operator $W : \mathfrak{g} \to \mathfrak{g}$ preserves the grading, so does its symmetric primitive P.

Main assumption: $\mathfrak{z} \supset \mathfrak{h}$

The central subalgebra $\mathfrak{z}$ contains some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (w.r.t.$[,]$)

Gradings and Main assumption

Definition

Let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a *grading* of a Lie algebra $(\mathfrak{g}, [,])$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i, j \in \Gamma$, Γ an abelian group. We say that a linear operator $W : \mathfrak{g} \to \mathfrak{g}$ *preserves* the grading if $W\mathfrak{g}_i \subset \mathfrak{g}_i$ for any $i \in \Gamma$.

Theorem

Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure and let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a grading. Then, if the principal operator $W : \mathfrak{g} \to \mathfrak{g}$ preserves the grading, so does its symmetric primitive P.

Main assumption: $\mathfrak{z} \supset \mathfrak{h}$

The central subalgebra $\mathfrak{z}$ contains some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (w.r.t.$[,]$)

Gradings and Main assumption

Definition

Let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a *grading* of a Lie algebra $(\mathfrak{g}, [,])$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i, j \in \Gamma$, Γ an abelian group. We say that a linear operator $W : \mathfrak{g} \to \mathfrak{g}$ *preserves* the grading if $W\mathfrak{g}_i \subset \mathfrak{g}_i$ for any $i \in \Gamma$.

Theorem

Let $(\mathfrak{g}, [,], [,]')$ be a semisimple bi-Lie structure and let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a grading. Then, if the principal operator $W : \mathfrak{g} \to \mathfrak{g}$ preserves the grading, so does its symmetric primitive P.

Main assumption: $\mathfrak{z} \supset \mathfrak{h}$

The central subalgebra $\mathfrak{z}$ contains some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (w.r.t.$[,]$)

Gradings and Main assumption

Theorem

The main assumption $\mathfrak{z} \supset \mathfrak{h}$ is equivalent to the following two conditions

- The principal operator $W \in \mathrm{End}(\mathfrak{g})$ preserves the grading

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

  related to the root decomposition with respect to the Cartan subalgebra $\mathfrak{h}$. In other words for some $\omega_\alpha \in \mathbb{C}$

$$W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}.$$

- The operator $W|_{\mathfrak{h}}$ is diagonalizable.

Consequences of the Main assumption I

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha}, \ P|_{\mathfrak{z}} = 0.$ *Then* $\mathfrak{z}$ *is a reductive in* $\mathfrak{g}$ *Lie subalgebra and for any root* α

- *there exist two times* $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$. *They are the solutions of the quadratic equation* $(t-\omega_\alpha)(t-\omega_{-\alpha}) - 2\pi_\alpha = 0$. *Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$, *then* $T_\alpha = T_{-\alpha}$.
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$, *where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$.
- $(W - t_{1,\alpha}I)(W - t_{2,\alpha}I)H_\alpha = 0$, *here* $H_\alpha \in \mathfrak{h}, \alpha \in R$, *is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$.

Consequently, $W|_{\mathfrak{h}}$ *is* admissible *in the following sense: for any root* α *the vector* $H_\alpha \in \mathfrak{h}$ *is either an eigenvector of* W, *or a sum of two eigenvectors of* W.

Consequences of the Main assumption I

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha},\ P|_{\mathfrak{z}} = 0$. *Then* $\mathfrak{z}$ *is a reductive in* $\mathfrak{g}$ *Lie subalgebra and for any root* α

- *there exist two times* $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$. *They are the solutions of the quadratic equation* $(t - \omega_\alpha)(t - \omega_{-\alpha}) - 2\pi_\alpha = 0$. *Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$, *then* $T_\alpha = T_{-\alpha}$.
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$, *where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$.
- $(W - t_{1,\alpha}I)(W - t_{2,\alpha}I)H_\alpha = 0$, *here* $H_\alpha \in \mathfrak{h}, \alpha \in R$, *is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$.

Consequently, $W|_{\mathfrak{h}}$ *is* admissible *in the following sense: for any root* α *the vector* $H_\alpha \in \mathfrak{h}$ *is either an eigenvector of* W*, or a sum of two eigenvectors of* W*.*

Consequences of the Main assumption I

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha},\ P|_{\mathfrak{z}} = 0$. *Then* $\mathfrak{z}$ *is a reductive in* $\mathfrak{g}$ *Lie subalgebra and for any root* α

- *there exist two* times $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$. *They are the solutions of the quadratic equation* $(t - \omega_\alpha)(t - \omega_{-\alpha}) - 2\pi_\alpha = 0$. *Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$, *then* $T_\alpha = T_{-\alpha}$.
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$, *where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$.
- $(W - t_{1,\alpha}I)(W - t_{2,\alpha}I)H_\alpha = 0$, *here* $H_\alpha \in \mathfrak{h}, \alpha \in R$, *is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$.

Consequently, $W|_{\mathfrak{h}}$ *is* admissible *in the following sense: for any root* α *the vector* $H_\alpha \in \mathfrak{h}$ *is either an eigenvector of* W*, or a sum of two eigenvectors of* W*.*

Consequences of the Main assumption I

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha},\ P|_{\mathfrak{z}} = 0$. *Then* $\mathfrak{z}$ *is a reductive in* $\mathfrak{g}$ *Lie subalgebra and for any root* α

- *there exist two* times $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$. *They are the solutions of the quadratic equation* $(t - \omega_\alpha)(t - \omega_{-\alpha}) - 2\pi_\alpha = 0$. *Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$, *then* $T_\alpha = T_{-\alpha}$.
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$, *where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$.
- $(W - t_{1,\alpha}I)(W - t_{2,\alpha}I)H_\alpha = 0$, *here* $H_\alpha \in \mathfrak{h}, \alpha \in R$, *is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$.

Consequently, $W|_{\mathfrak{h}}$ *is admissible in the following sense: for any root* α *the vector* $H_\alpha \in \mathfrak{h}$ *is either an eigenvector of* W*, or a sum of two eigenvectors of* W*.*

Consequences of the Main assumption I

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha},\ P|_{\mathfrak{z}} = 0$. *Then* $\mathfrak{z}$ *is a reductive in* $\mathfrak{g}$ *Lie subalgebra and for any root* α

- *there exist two* times $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$. *They are the solutions of the quadratic equation* $(t - \omega_\alpha)(t - \omega_{-\alpha}) - 2\pi_\alpha = 0$. *Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$, *then* $T_\alpha = T_{-\alpha}$.
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$, *where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$.
- $(W - t_{1,\alpha} I)(W - t_{2,\alpha} I)H_\alpha = 0$, *here* $H_\alpha \in \mathfrak{h}, \alpha \in R$, *is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$.

Consequently, $W|_{\mathfrak{h}}$ *is* admissible *in the following sense: for any root* α *the vector* $H_\alpha \in \mathfrak{h}$ *is either an eigenvector of* W, *or a sum of two eigenvectors of* W.

Consequences of the Main assumption I

Theorem

Recall $W|_{\mathfrak{g}_\alpha} = \omega_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, W\mathfrak{h} \subset \mathfrak{h}, P|_{\mathfrak{g}_\alpha} = \pi_\alpha \mathrm{Id}_{\mathfrak{g}_\alpha}, \pi_\alpha = \pi_{-\alpha},\ P|_{\mathfrak{z}} = 0$*. Then* $\mathfrak{z}$ *is a reductive in* $\mathfrak{g}$ *Lie subalgebra and for any root* α

- *there exist two* times $t_{1,\alpha}, t_{2,\alpha}$ *(possibly equal) such that* $\mathfrak{g}_\alpha \subset \ker B^{t_{1,\alpha}} \cap \ker B^{t_{2,\alpha}}$*. They are the solutions of the quadratic equation* $(t-\omega_\alpha)(t-\omega_{-\alpha}) - 2\pi_\alpha = 0$*. Moreover, if* $T_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$*, then* $T_\alpha = T_{-\alpha}$*.*
- $\sigma_\alpha = (1/2)(t_{1,\alpha} + t_{2,\alpha}), \kappa_\alpha = \pm\sqrt{((t_{1,\alpha} - t_{2,\alpha})/2)^2 - 2\pi_\alpha}$*, where* $\sigma_\alpha := (1/2)(\omega_\alpha + \omega_{-\alpha}), \kappa_\alpha := (1/2)(\omega_\alpha - \omega_{-\alpha})$*.*
- $(W - t_{1,\alpha}I)(W - t_{2,\alpha}I)H_\alpha = 0$*, here* $H_\alpha \in \mathfrak{h}, \alpha \in R$*, is such that* $B(H_\alpha, H) = \alpha(H)$ *for any* $H \in \mathfrak{h}$*.*

Consequently, $W|_{\mathfrak{h}}$ *is* admissible *in the following sense: for any root* α *the vector* $H_\alpha \in \mathfrak{h}$ *is either an eigenvector of* W*, or a sum of two eigenvectors of* W*.*

Consequences of the Main assumption II

Theorem

Let α, β, γ be roots such that $\alpha + \beta + \gamma = 0$. Then only the following possibilities can occur ("times selection rules"):

1. *either there exist $t_1, t_2, t_3 \in \mathbb{C}$ such that*

$$T_\alpha = \{t_1, t_2\}, T_\beta = \{t_2, t_3\}, T_\gamma = \{t_3, t_1\};$$

2. *or there exist $t_1, t_2 \in \mathbb{C}$ such that*

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}, t_1 \neq t_2,$$

Moreover, in Case 1 the following equality holds:

$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0$$

and in Case 2:

$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm(t_1 - t_2)/2.$$

Consequences of the Main assumption III

SS bi-Lie structures $\overset{1:1}{\Longleftrightarrow}$ $(U, \mathcal{T})$, $U : \mathfrak{h} \to \mathfrak{h}$ admissible, $\mathcal{T}$ a pair diagram

Pair diagrams

$\mathcal{T} = \{T_\alpha\}_{\alpha \in R}$, $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$, $t_{i,\alpha} \in \mathbb{C}$ obeying the "times selection rules"

Examples:

$t_1 t_3$

$t_1 t_3$ $t_2 t_3$,

$t_1 t_2$ $t_2 t_3$ $t_3 t_3$

$t_1 t_2$

$t_1 t_2$ $t_1 t_2$.

$t_1 t_1$ $t_1 t_2$ $t_1 t_2$

Theorem $\Longrightarrow$ two classes of pair diagrams, I and II

Assume that there exist roots α, β, γ such that $\alpha + \beta + \gamma = 0$ and

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}$$

for some $t_1, t_2, t_1 \neq t_2$. Then $T_\delta = \{t_1, t_2\}, \{t_1, t_1\}$ or $\{t_2, t_2\}$ for any δ.

Consequences of the Main assumption III

SS bi-Lie structures $\overset{1:1}{\Longleftrightarrow} (U, \mathcal{T})$, $U : \mathfrak{h} \to \mathfrak{h}$ admissible, $\mathcal{T}$ a pair diagram

Pair diagrams

$\mathcal{T} = \{T_\alpha\}_{\alpha \in R}$, $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$, $t_{i,\alpha} \in \mathbb{C}$ obeying the "times selection rules"

Examples:

$$\begin{matrix} & & t_1 t_3 & & \\ & t_1 t_3 & & t_2 t_3 & \\ t_1 t_2 & & t_2 t_3 & & t_3 t_3 \end{matrix} \quad , \quad \begin{matrix} & & t_1 t_2 & & \\ & t_1 t_2 & & t_1 t_2 & \\ t_1 t_1 & & t_1 t_2 & & t_1 t_2 \end{matrix} \quad .$$

Theorem $\Longrightarrow$ two classes of pair diagrams, I and II

Assume that there exist roots α, β, γ such that $\alpha + \beta + \gamma = 0$ and

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}$$

for some $t_1, t_2, t_1 \neq t_2$. Then $T_\delta = \{t_1, t_2\}, \{t_1, t_1\}$ or $\{t_2, t_2\}$ for any δ.

Consequences of the Main assumption III

SS bi-Lie structures $\overset{1:1}{\Longleftrightarrow}$ $(U, \mathcal{T})$, $U : \mathfrak{h} \to \mathfrak{h}$ admissible, $\mathcal{T}$ a pair diagram

Pair diagrams

$\mathcal{T} = \{T_\alpha\}_{\alpha \in R}$, $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$, $t_{i,\alpha} \in \mathbb{C}$ obeying the "times selection rules"

Examples:

$$\begin{array}{ccccc} & & t_1 t_3 & & \\ & t_1 t_3 & & t_2 t_3 & \\ t_1 t_2 & & t_2 t_3 & & t_3 t_3 \end{array} \quad , \quad \begin{array}{ccccc} & & t_1 t_2 & & \\ & t_1 t_2 & & t_1 t_2 & \\ t_1 t_1 & & t_1 t_2 & & t_1 t_2 \end{array} \quad .$$

Theorem $\Longrightarrow$ two classes of pair diagrams, I and II

Assume that there exist roots α, β, γ such that $\alpha + \beta + \gamma = 0$ and

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}$$

for some $t_1, t_2, t_1 \neq t_2$. Then $T_\delta = \{t_1, t_2\}, \{t_1, t_1\}$ or $\{t_2, t_2\}$ for any δ.

Consequences of the Main assumption III

SS bi-Lie structures $\overset{1:1}{\Longleftrightarrow} (U, \mathcal{T})$, $U : \mathfrak{h} \to \mathfrak{h}$ admissible, $\mathcal{T}$ a pair diagram

Pair diagrams

$\mathcal{T} = \{T_\alpha\}_{\alpha \in R}$, $T_\alpha = \{t_{1,\alpha}, t_{2,\alpha}\}$, $t_{i,\alpha} \in \mathbb{C}$ obeying the "times selection rules"

Examples:

$$\begin{array}{ccccc} & & t_1 t_3 & & \\ & t_1 t_3 & & t_2 t_3 & \\ t_1 t_2 & & t_2 t_3 & & t_3 t_3 \end{array} , \qquad \begin{array}{ccccc} & & t_1 t_2 & & \\ & t_1 t_2 & & t_1 t_2 & \\ t_1 t_1 & & t_1 t_2 & & t_1 t_2 \end{array} .$$

Theorem $\Longrightarrow$ two classes of pair diagrams, I and II

Assume that there exist roots α, β, γ such that $\alpha + \beta + \gamma = 0$ and

$$T_\alpha = T_\beta = T_\gamma = \{t_1, t_2\}$$

for some $t_1, t_2, t_1 \neq t_2$. Then $T_\delta = \{t_1, t_2\}, \{t_1, t_1\}$ or $\{t_2, t_2\}$ for any δ.

Examples of bi-Lie structures of Class I

Example

$R = \mathfrak{d}_n$, roots $\pm\epsilon_i \pm \epsilon_j (1 \leq i < j \leq n), U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i\pm\epsilon_j} := \{t_i, t_j\}$ (KP1, $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n)$).

$R = \mathfrak{b}_n$, roots $\pm\epsilon_i (1 \leq i \leq n), \pm\epsilon_i \pm \epsilon_j (1 \leq i < j \leq n) U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i\pm\epsilon_j} := \{t_i, t_j\}$, $T_{\pm\epsilon_i} := \{t_i, (t_{n+1})\}$ (KP1, $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n, t_{n+1})$).

$R = \mathfrak{c}_n$, roots $\pm 2\epsilon_i (1 \leq i \leq n), \pm\epsilon_i \pm \epsilon_j (1 \leq i < j \leq n) U\epsilon_i = t_i\epsilon_i$, $T_{\pm\epsilon_i\pm\epsilon_j} := \{t_i, t_j\}$, $T_{\pm 2\epsilon_i} := \{t_i, t_i\}$ (KP2,$A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n)$).

Examples of bi-Lie structures of Class I

$R = \mathfrak{a}_n$, root basis $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_n = \epsilon_n - \epsilon_{n+1}$.

a) Put $w_n := a\alpha_n, w_{n-1} := w_n + \alpha_{n-1}, \ldots, w_1 := w_2 + \alpha_1$, where $a \neq 0, 1$, $U(w_i) := t_i w_i$, $T_{\pm(\epsilon_i - \epsilon_j)} := \{t_i t_j\}$, if $i < j < n+1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i t_n\}$ (new).

$t_1 t_3$

$t_1 t_3$ $t_2 t_3$

$t_1 t_2$ $t_2 t_3$ $t_3 t_3$

b) Put $a = 1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i (t_{n+1})\}$

$t_1(t_4)$

$t_1 t_3$ $t_2(t_4)$

$t_1 t_2$ $t_2 t_3$ $t_3(t_4)$

(new, corresponds to $WX = (1/2)(L_A + R_A)X - \mathrm{Tr}((1/2)(L_A + R_A)X)B$, where $X \in \mathfrak{sl}(n+1), A = \mathrm{diag}(t_1, t_2, \ldots, t_{n+1}), B = \mathrm{diag}(0, 0, \ldots, 0, 1)$).

Conjecture

Any bi-Lie structure of Class I is from the list above.

($\Leftarrow$ classification of specific $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$-gradings)

Examples of bi-Lie structures of Class I

$R = \mathfrak{a}_n$, root basis $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_n = \epsilon_n - \epsilon_{n+1}$.

a) Put $w_n := a\alpha_n, w_{n-1} := w_n + \alpha_{n-1}, \ldots, w_1 := w_2 + \alpha_1$, where $a \neq 0, 1$, $U(w_i) := t_i w_i$, $T_{\pm(\epsilon_i - \epsilon_j)} := \{t_i t_j\}$, if $i < j < n+1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i t_n\}$ (new).

$t_1 t_3$

$t_1 t_3$ $\quad$ $t_2 t_3$

$t_1 t_2$ $\quad$ $t_2 t_3$ $\quad$ $t_3 t_3$

b) Put $a = 1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i(t_{n+1})\}$

$t_1(t_4)$

$t_1 t_3$ $\quad$ $t_2(t_4)$

$t_1 t_2$ $\quad$ $t_2 t_3$ $\quad$ $t_3(t_4)$

(new, corresponds to $WX = (1/2)(L_A + R_A)X - \mathrm{Tr}((1/2)(L_A + R_A)X)B$, where $X \in \mathfrak{sl}(n+1), A = \mathrm{diag}(t_1, t_2, \ldots, t_{n+1}), B = \mathrm{diag}(0, 0, \ldots, 0, 1)$).

Conjecture

Any bi-Lie structure of Class I is from the list above.

($\Leftarrow$ classification of specific $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$-gradings)

Examples of bi-Lie structures of Class I

$R = \mathfrak{a}_n$, root basis $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_n = \epsilon_n - \epsilon_{n+1}$.

a) Put $w_n := a\alpha_n, w_{n-1} := w_n + \alpha_{n-1}, \ldots, w_1 := w_2 + \alpha_1$, where $a \neq 0, 1$, $U(w_i) := t_i w_i$, $T_{\pm(\epsilon_i - \epsilon_j)} := \{t_i t_j\}$, if $i < j < n+1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i t_n\}$ (new).

$t_1 t_3$

$t_1 t_3$ $\quad$ $t_2 t_3$

$t_1 t_2$ $\quad$ $t_2 t_3$ $\quad$ $t_3 t_3$

b) Put $a = 1$ and $T_{\pm(\epsilon_i - \epsilon_{n+1})} = \{t_i(t_{n+1})\}$

$t_1(t_4)$

$t_1 t_3$ $\quad$ $t_2(t_4)$

$t_1 t_2$ $\quad$ $t_2 t_3$ $\quad$ $t_3(t_4)$

(new, corresponds to $WX = (1/2)(L_A + R_A)X - \mathrm{Tr}((1/2)(L_A + R_A)X)B$, where $X \in \mathfrak{sl}(n+1), A = \mathrm{diag}(t_1, t_2, \ldots, t_{n+1}), B = \mathrm{diag}(0, 0, \ldots, 0, 1)$).

Conjecture

Any bi-Lie structure of Class I is from the list above.

($\Leftarrow$ classification of specific $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$-gradings)

Examples of bi-Lie structures of Class II

Example 1

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$ related to an *inner* automorphism of n-th order, $n > 2$, and $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ (GS1 with *inner* automorphism of n-th order, $n > 2$).

Example 2

Let $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_0$ is a parabolic subalgebra and $\tilde{\mathfrak{g}}_1$ its "complement" $W|_{\tilde{\mathfrak{g}}_i} = \omega_i \mathrm{Id}_{\tilde{\mathfrak{g}}_i}$, ω_i arbitrary (P).

Theorem

Any Example 2 is isomorphic to one of the Examples 1 (for which $\mathfrak{g}_0$ is a Levi subalgebra)

Theorem

Any bi-Lie structure of Class II for $\mathfrak{g} = \mathfrak{a}_n$ is a modification of Example 1 (belongs to GS1).

Examples of bi-Lie structures of Class II

Example 1

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$ related to an *inner* automorphism of n-th order, $n > 2$, and $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ (GS1 with *inner* automorphism of n-th order, $n > 2$).

Example 2

Let $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_0$ is a parabolic subalgebra and $\tilde{\mathfrak{g}}_1$ its "complement" $W|_{\tilde{\mathfrak{g}}_i} = \omega_i \mathrm{Id}_{\tilde{\mathfrak{g}}_i}$, ω_i arbitrary (P).

Theorem

Any Example 2 is isomorphic to one of the Examples 1 (for which $\mathfrak{g}_0$ is a Levi subalgebra)

Theorem

Any bi-Lie structure of Class II for $\mathfrak{g} = \mathfrak{a}_n$ is a modification of Example 1 (belongs to GS1).

Examples of bi-Lie structures of Class II

Example 1

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$ related to an *inner* automorphism of n-th order, $n > 2$, and $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ (GS1 with *inner* automorphism of n-th order, $n > 2$).

Example 2

Let $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_0$ is a parabolic subalgebra and $\tilde{\mathfrak{g}}_1$ its "complement" $W|_{\tilde{\mathfrak{g}}_i} = \omega_i \mathrm{Id}_{\tilde{\mathfrak{g}}_i}$, ω_i arbitrary (P).

Theorem

Any Example 2 is isomorphic to one of the Examples 1 (for which $\mathfrak{g}_0$ is a Levi subalgebra)

Theorem

Any bi-Lie structure of Class II for $\mathfrak{g} = \mathfrak{a}_n$ is a modification of Example 1 (belongs to GS1).

Examples of bi-Lie structures of Class II

Example 1

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$ related to an *inner* automorphism of n-th order, $n > 2$, and $W|_{\mathfrak{g}_i} = i\mathrm{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$ (GS1 with *inner* automorphism of n-th order, $n > 2$).

Example 2

Let $\mathfrak{g} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1$, where $\tilde{\mathfrak{g}}_0$ is a parabolic subalgebra and $\tilde{\mathfrak{g}}_1$ its "complement" $W|_{\tilde{\mathfrak{g}}_i} = \omega_i \mathrm{Id}_{\tilde{\mathfrak{g}}_i}$, ω_i arbitrary (P).

Theorem

Any Example 2 is isomorphic to one of the Examples 1 (for which $\mathfrak{g}_0$ is a Levi subalgebra)

Theorem

Any bi-Lie structure of Class II for $\mathfrak{g} = \mathfrak{a}_n$ is a modification of Example 1 (belongs to GS1).

Bi-Lie structures of Class II

Conjecture

Any bi-Lie structure of Class II is a modification of Example 1 (belongs to GS1).

Perspectives

- Classification without Main assumption
- Nonsemisimple algebras
- Invariant Nijenhuis and "weak Nijenhuis" (1,1)-tensors on homogeneous spaces
- Clarification of relations with classical R-matrix formalism

Bi-Lie structures of Class II

Conjecture

Any bi-Lie structure of Class II is a modification of Example 1 (belongs to GS1).

Perspectives

- Classification without Main assumption
- Nonsemisimple algebras
- Invariant Nijenhuis and "weak Nijenhuis" (1,1)-tensors on homogeneous spaces
- Clarification of relations with classical R-matrix formalism

Bi-Lie structures of Class II

Conjecture

Any bi-Lie structure of Class II is a modification of Example 1 (belongs to GS1).

Perspectives

- Classification without Main assumption
- Nonsemisimple algebras
- Invariant Nijenhuis and “weak Nijenhuis” (1,1)-tensors on homogeneous spaces
- Clarification of relations with classical R-matrix formalism

Selected bibliography

- Kantor, I. L., and D. B. Persits, *On closed pencils of linear Poisson brackets*, (Russian) in: IXth All-Union Geometric Conference, Kishinev: Shtiintsa, 1988, 141.
- Holod, P. I., *Hidden symmetry of the Landau-Lifshitz equation, its higher analogues and dual equation for asymmetric chiral field*, (Russian) Teoret. Matem. Fiz. 70 (1987), 1829, English Translation: Theoret. Math. Phys. 70 (1987), 1119.
- Golubchik, I. Z., and V. V. Sokolov, *Compatible Lie brackets and integrable equations of the principal chiral model type*, (Russian) Funkts. Anal. Prilozh. 36 (2002), 919, English Translation: Funct. Anal. Appl. 36 (2002), 172181.
- Odesskii, A., and V. Sokolov, *Algebraic structures connected with pairs of compatible associative algebras*, Int. Math. Res. Not. (2006), 35 pp., Art. ID 43734.

Many thanks for your attention!

Selected bibliography

- Kantor, I. L., and D. B. Persits, *On closed pencils of linear Poisson brackets*, (Russian) in: IXth All-Union Geometric Conference, Kishinev: Shtiintsa, 1988, 141.
- Holod, P. I., *Hidden symmetry of the Landau-Lifshitz equation, its higher analogues and dual equation for asymmetric chiral field*, (Russian) Teoret. Matem. Fiz. 70 (1987), 1829, English Translation: Theoret. Math. Phys. 70 (1987), 1119.
- Golubchik, I. Z., and V. V. Sokolov, *Compatible Lie brackets and integrable equations of the principal chiral model type*, (Russian) Funkts. Anal. Prilozh. 36 (2002), 919, English Translation: Funct. Anal. Appl. 36 (2002), 172181.
- Odesskii, A., and V. Sokolov, *Algebraic structures connected with pairs of compatible associative algebras*, Int. Math. Res. Not. (2006), 35 pp., Art. ID 43734.

Many thanks for your attention!