A note on actions of some monoids

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Subject of interest

- $\mathrm{T}M$ - an important, and in some sense universal, example of a vector bundle. Any vector bundle is a subbundle of the tangent bundle:
$$E \subset \mathrm{V}E \subset \mathrm{T}E,\ v \mapsto [t \mapsto tv]$$
- On any vector bundle acts the monoid $(\mathbb{R}, \cdot)$ (the multiplication of a vector by a real number)

Theorem (J. Grabowski, M.R.(2009))

Vector bundles are just manifolds equipped with $(\mathbb{R}, \cdot)$-action which satisfy certain non-singularity condition.

In particular, the addition on a vector bundle is completely determined by the multiplication by reals (yet the smoothness of this multiplication at $0 \in \mathbb{R}$ is essential).

Theorem (Euler)

Any smooth homogenous function $f : \mathbb{R}^n \to \mathbb{R}$ has to be linear.

Subject of interest

Studies on smooth actions of the multiplicative monoid of real numbers, $(\mathbb{R}, \cdot)$, on smooth manifolds lead to the concept of a *graded bundle*, a natural generalisation of the notion of a vector bundle. The fundamental example is the k-th order tangent bundle, $\mathrm{T}^k M$, $k \geq 1$, consisting of k-th order tangency classes $[\gamma]_k$ (k-velocities) of curves γ in M. For $k > 1$, $\mathrm{T}^k M$ is no longer a vector bundle.

Example ($\mathrm{T}^k M$)

Local coordinates (x^a) on M induce *adapted coordinates* $(x^a, \dot{x}^a, \ddot{x}^a, \ldots)$ for $\mathrm{T}^k M$ ($\gamma : \mathbb{R} \to M$):

$$\gamma^a(t) = x^a(\gamma(t)) = x^a(\gamma(0)) + \dot{x}^a t + \ddot{x}^a t^2/2! + \ldots + x^{a,(k)} t^k/k! + o(t^k).$$

They are naturally graded. On $\mathrm{T}^2 M$ they transform as

$$x^{a'} = x^{a'}(x), \underbrace{\dot{x}^{a'}}_{w=1} = \frac{\partial x^{a'}}{\partial x^b} \underbrace{\dot{x}^b}_{w=1},$$

$$\underbrace{\ddot{x}^{a'}}_{w=2} = \frac{\partial x^{a'}}{\partial x^b} \underbrace{\ddot{x}^b}_{w=2} + \frac{\partial^2 x^{a'}}{\partial x^b \partial x^c} \underbrace{\dot{x}^b \dot{x}^c}_{1+1=2}.$$

In general, a gradation of coordinates leads to the concept of a *graded bundle*.

Subject of interest

It is natural to ask

(Q1) What are geometric structures naturally related with smooth monoid actions on manifolds for monoids $\mathcal{G}$ other than $(\mathbb{R}, \cdot)$?

(Q2) How to characterize smooth actions of the multiplicative reals $(\mathbb{R}, \cdot)$ on supermanifolds?

Of course it is hopeless to discuss (Q1) for an arbitrary monoid $\mathcal{G}$. Therefore we concentrate our attention on several special cases, all being natural generalizations of the monoid $(\mathbb{R}, \cdot)$ of multiplicative reals:

- $\mathcal{G} = (\mathbb{C}, \cdot)$ is the multiplicative monoid of complex numbers;
- $\mathcal{G} = \mathcal{G}_2$, where $\mathcal{G}_k$ is the monoid of the k^{th}-jets of punctured maps $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ (note that $(\mathbb{R}, \cdot)$ can be viewed as the monoid of the 1^{st}-jets of such maps);
- $\mathcal{G} = \mathsf{M}_2(\mathbb{R})$ is the monoid of 2 by 2 real matrices.

Subject of interest

Problem (Q2) has a rather expected answer yet it is not an obvious result. The answer is that supermanifolds equipped with a smooth action of the monoid $(\mathbb{R}, \cdot)$ are *graded bundles in the category of supermanifold*. In 2005 P. Severa states (without a proof) that "an N-manifold (shorthand for 'non-negatively graded manifold') is a supermanifold with an action of the multiplicative semigroup $(\mathbb{R}, \cdot)$ such that -1 acts as the parity operator", which is a statement slightly weaker than our result. Nevertheless, a rigorous proof seems to be missing in the literature, and we fill this gap.

Let $\mathrm{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k_{\geq 0}$, I a set of cardinality $|\mathrm{d}| := d_1 + \ldots + d_k$, and let $I \ni \alpha \mapsto \mathbf{w}(\alpha) \in \mathbb{Z}_{>0}$, be a map such that $d_i = \#\{\alpha : \mathbf{w}(\alpha) = i\}$ for $1 \leq i \leq k$.

A *graded space of rank* d over a field $\mathbb{K}$ is a set W equipped with an equivalence class of *graded coordinates* of rank d. By definition, a *system of rank* d *graded coordinates* on W is a bijective map $(y^\alpha)_{\alpha \in I} : \mathrm{W} \xrightarrow{\simeq} \mathbb{K}^{|\mathrm{d}|}$, with *weight* $\mathbf{w}(\alpha) \in \mathbb{N}$ assigned to each function y^α, $\alpha \in I$.

Two systems of graded coordinates, (y^α) and $(\underline{y}^\alpha)$ are *equivalent* if there exist constants $c^\alpha_{\alpha_1 \ldots \alpha_j} \in \mathbb{K}$, defined for indices such that $\mathbf{w}(\alpha) = \mathbf{w}(\alpha_1) + \ldots + \mathbf{w}(\alpha_j)$, such that

$$\underline{y}^\alpha = \sum_{\substack{j=1,2,\ldots \\ \mathbf{w}(\alpha)=\mathbf{w}(\alpha_1)+\ldots+\mathbf{w}(\alpha_j)}} c^\alpha_{\alpha_1 \ldots \alpha_j} y^{\alpha_1} \ldots y^{\alpha_k}. \tag{1}$$

The highest non-zero coordinate weight is called the *degree* of a graded space W. Note: a graded space of rank (d_1) is just a vector space of dimension d_1.

Homotheties in a graded space

The map

$$(y^\alpha) \mapsto (t^{\mathbf{w}(\alpha)} y^\alpha),\, t \in \mathbb{K}$$

defines a canonical action of the monoid $(\mathbb{K}, \cdot)$ referred as the *action by homotheties* on a graded space W.

Real/complex/holomorphic graded bundles

Let $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$). A graded space W over the field $\mathbb{K}$ is naturally a smooth (resp. complex) manifold in which any system of graded coordinates of W is a smooth (resp. holomorphic) map.
The notions of smooth / complex / holomorphic vector bundles have their clear 'graded' analogs called smooth/complex/holomorphic graded bundles.
Any smooth (resp. complex / holomorphic) graded bundle is equipped with a canonical action by homotheties of the monoid $(\mathbb{R}, \cdot)$ (resp. $(\mathbb{C}, \cdot)$).
The vector field Δ_M on M associated with the flow $(t, p) \mapsto h_{e^t}(p)$ is called the *weight vector field* of M. In graded coordinates (y^α),

$$\Delta_M = \sum_\alpha \mathbf{w}(\alpha) y^\alpha \partial_{y^\alpha}.$$

A smooth (resp. complex/holomorphic) *homogeneity structure* on a smooth (resp., smooth / complex) manifold M is a smooth (resp., smooth / holomorphic) action of the multiplicative monoid $(\mathbb{R}, \cdot)$ (resp. $(\mathbb{C}, \cdot)$ / $(\mathbb{C}, \cdot)$)

$$h : \mathbb{R} \times M \longrightarrow M \text{ (resp. } h : \mathbb{C} \times M \longrightarrow M\text{)}.$$

A *morphism* between two smooth (resp. complex/holomorphic) homogeneity structures (M_1, h^1) and (M_2, h^2) is a smooth (resp. smooth/holomorphic) map $\Phi : M_1 \to M_2$ intertwining the actions h^1 and h^2, i.e.,

$$\Phi(h^1_\xi(p)) = h^2_\xi(\Phi(p)),$$

for every $p \in M_1$ and every $\xi \in \mathbb{C}$, where $h_\xi = h(\xi, \cdot)$. Clearly, smooth (resp. complex / holomorphic) homogeneity structures with their morphisms form a *category*.

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Homogeneity structures

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Homogeneity structures vs. graded bundles

Theorem (J. Grabowski, M. Rotkiewicz (2011))

Let (M, h) be a (connected) smooth homogeneity structure. Then $h_0 : M \to M_0$ is a graded bundle with homotheties defined by h.

Idea of proof. For large enough k the canonical map $\phi : M \to \mathrm{T}^k M$ sending $p \in M$ to the class of the curve $t \mapsto h(t, p)$ is an embedding. Moreover, the image of M inherits from $\mathrm{T}^k M$ a structure of a smooth graded bundle.

In holomorphic setting we follow a similar idea. A higher order tangent bundle $\mathrm{T}^k M$ is replaced with the space $\mathsf{J}^k M$ of k^{th}-jets of holomorphic curves in a complex manifold M which turns out to be a universal example of a holomorphic graded bundle.

Theorem (M. Jóźwikowski, M. Rotkiewicz (2016))

The categories of (connected) holomorphic graded bundles and (connected) holomorphic homogeneity structures are equivalent.

Nice complex homogeneity structures

Example

Consider $M = \mathbb{C}$ with a standard coordinate $z : M \to \mathbb{C}$ and define a smooth multiplicative action $h : \mathbb{C} \times M \to M$ by the formula $h(\xi, z) = |\xi|^2 \xi\, z$. We claim that h is not a homothety action related with any complex graded bundle structure on M. Hint: consider the action of $\exp(2\pi\sqrt{-1}/3)$.

To formulate forthcoming theorem we recall that the *core* of a graded bundle of degree k is defined by vanishing of all graded fiber coordinates except for coordinates of highest weight.

Nice complex homogeneity structures

In order to characterize those complex homogeneity structures (M, h) that comes from complex graded bundles we introduce the notion of *nice* complex homogeneity structure. First of all, $(M, h|_{\mathbb{R}\times M})$ is a smooth graded bundle, while $h_\xi : M \to M$ is a morphism of smooth graded bundles, for any $\xi \in \mathbb{C}$ (Why?). In particular, h_ξ gives a vector bundle map on the cores of graded bundles.

Definition

Let $h : \mathbb{C} \times E^k \to E^k$ be a complex homogeneity structure and $\tau = h_0 : E^k \to M$ be the (real) graded bundle, say of degree k, associated with $h|_{\mathbb{R}}$. We say that the action h is *nice* if $h_{\varepsilon_{2j}}$ on the core bundle of E^j acts as minus the identity for each $j = 1, 2, \ldots, k$, where $\varepsilon_{2j} = \exp 2\pi/2j$.

Nice complex homogeneity structures

Theorem (M. Jóźwikowski, M. Rotkiewicz (2016))

The categories of (connected) nice complex graded bundles and (connected) complex homogeneity structures are equivalent.

Let

$$\mathcal{G}_k := \{[\phi]_k \mid \phi : \mathbb{R} \to \mathbb{R},\ \phi(0) = 0\} \simeq \mathbb{R}^k,$$
$$[t \mapsto a_1 t + \frac{a_2}{2} t^2 + \ldots + \frac{a_k}{k!} t^k + o(t^k)] \mapsto (a_1, \ldots, a_k)$$

denotes the monoid of the k^{th}-jets of punctured maps $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $[\phi]_k \cdot [\psi]_k := [\phi \circ \psi]_k$.
Note:

- $\mathcal{G}_1 \simeq (\mathbb{R}, \cdot)$.
- There is no embedding of $\mathcal{G}_k$ into a group.

Goal: to study smooth $\mathcal{G}_k$-actions.

Left and right monoid actions.

Let $\mathcal{G}$ be an arbitrary monoid. Note that any left $\mathcal{G}$-action $(g, p) \mapsto g.p$, gives rise to a right $\mathcal{G}^{\mathrm{op}}$-action on the same manifold M given by the formula $p.g = g.p$. However, unlike the case of Lie groups actions, in general, there is no canonical correspondence between left and right $\mathcal{G}$-actions.

Example

The natural composition of maps gives a well defined right $\mathcal{G}_k$-action on the manifold $\mathrm{T}^k M$:

$$[\gamma]_k \circ [\phi]_k \mapsto [\gamma \circ \phi]_k,$$

for $\phi : \mathbb{R} \to \mathbb{R}$, $\gamma : \mathbb{R} \to M$, $\phi(0) = 0$.

Left and right monoid actions.

Example

Following Tulczyjew's notation, the higher cotangent space to a manifold M at a point $p \in M$, denoted by $\mathrm{T}^{k*}_p M$, consists of k^{th}-jets at $p \in M$ of functions $f : M \to \mathbb{R}$ such that $f(p) = 0$. The higher cotangent bundle $\mathrm{T}^{k*} M = \bigcup_{p \in M} \mathrm{T}^{k*}_p M$ is a vector bundle over M. The natural composition of maps gives rise to a left $\mathcal{G}_k$-action on $\mathrm{T}^{k*} M$,

$$[\phi]_k \circ [(f, p)]_k \mapsto [(\phi \circ f, p)]_k,$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is as before, and $f : M \to \mathbb{R}$ is such that $f(p) = 0$.

We shall concentrate now on actions of $\mathcal{G}_2$.
The monoid $\mathcal{G}_2$ contains two submonoids:

- the multiplicative reals $(\mathbb{R}, \cdot) \simeq \{(a, 0) : a \in \mathbb{R}\}$,
- the additive group $(\mathbb{R}, +) \simeq \{(1, b) : b \in \mathbb{R}\}$.

The action of $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$ on M makes M a graded bundle while the $(\mathbb{R}, +)$- action is a flow of some vector field. It is crucial to recognise the relation between this two structures. On infinitesimal level we get

Left and right actions of $\mathcal{G}_2$

Lemma

Every smooth right (respectively, left) action $H : M \times \mathcal{G}_2 \to M$ (resp., $H : \mathcal{G}_2 \times M \to M$) on a smooth manifold M provides M with:

- *a canonical graded bundle structure $\pi : M \to M_0 := H_{(0,0)}(M)$ induced by the action of the submonoid $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$,*
- *and a complete vector field $X \in \mathfrak{X}(M)$ of weight -1 (resp., $Y \in \mathfrak{X}(M)$ of weight $+1$) with respect to the above graded bundle structure on M.*

In other words, any $\mathcal{G}_2$-action provides M with two complete vector fields: the weight vector field Δ and another vector field X (respectively, Y), such that their Lie bracket satisfies

$$[\Delta, X] = -X, \quad (\textit{resp.}, [\Delta, Y] = Y) .$$

Infinitesimal data for the canonical $\mathcal{G}_2$-actions on $\mathrm{T}^2 M$ and $\mathrm{T}^{2*} M$

Example ($T^2 M$)

$$(x^i, \dot{x}^i, \ddot{x}^i).(a, b) = (x^i, a\dot{x}^i, a^2\ddot{x}^i + b\dot{x}^i)$$
$$\Delta = \dot{x}^i \partial_{\dot{x}^i} + 2\,\ddot{x}^i \partial_{\ddot{x}^i}$$
$$X = \dot{x}^i \partial_{\ddot{x}^i}.$$

Example ($T^{2*} M$)

$$(a, b).(p_i, p_{ij}) = (ap_i, ap_{ij} + bp_i\, p_j)$$
$$\Delta = p_i \partial_{p_i} + p_{ij} \partial_{p_{ij}}$$
$$Y = p_i p_j \partial_{p_{ij}}.$$

Our idea is to characterize right (resp. left) $\mathcal{G}_2$-actions in terms of a complete vector field X of weight -1 (resp., Y of weight +1) on a total space of a graded bundle. Such a data can always be integrated to an action of the Lie subgroup $\mathcal{G}_2^{\mathrm{inv}}$ of invertible elements of $\mathcal{G}_2$. Yet problems with extending this action on the whole $\mathcal{G}_2$ may appear.

In case the associated graded bundle $(E^k, H|_{(\mathbb{R},\cdot)})$ is of low degree we were able to explicitly characterise right $\mathcal{G}_2$-actions. For example, if the degree k is one (a vector bundle case) then the action of $(a, b) \in \mathcal{G}_2$ has to coincide with the action by homotheties of $a \in \mathbb{R}$. In degree $k = 2$ we have

Lemma

Right $\mathcal{G}_2$-actions of order 2 are in one-to-one correspondence with degree 2 graded bundles E^2 equipped with vector bundle morphisms $\phi : E^1 \to \check{E}^2$ covering id_M, where $\check{E}^2$ denotes the core of E^2:

$$(x^\alpha, \underbrace{y^i}_{w=1}, \underbrace{z^\mu}_{w=2}).(a, b) = (x^\alpha, a\, y^i, a^2\, z^\mu + b\, Q^\mu_i\, y^i).$$

We can fully characterise left $\mathcal{G}_2$-actions in case $k = 1$.

Lemma

Let $\tau : E \to M$ be a vector bundle. There is a one-to-one correspondence between smooth left $\mathcal{G}_2$-actions on E such that the multiplicative submonoid $(\mathbb{R}, \cdot) \subset \mathcal{G}_2$ acts by the homotheties of E and symmetric bi-linear operations $\bullet : E \times_M E \to E$ such that for any $v \in E$

$$v \bullet (v \bullet v) = 0 \ .$$

This correspondence is given by the following formula

$$(a, b).v = a\, v + b\, v \bullet v \ ,$$

where $(a, b) = [t \mapsto at + bt^2/2 + o(t^2)] \in \mathcal{G}_2$ and $v \in E$.

Lemma

Any $\mathrm{M}_2(\mathbb{R})$*-action on a manifold* M *give rise to a double graded bundle* (M, Δ_1, Δ_2) *equipped with two complete vector fields* X, Y *of weights* $(1, -1)$ *and* $(-1, 1)$ *respectively, such that* $[X, Y] = \Delta^1 - \Delta^2$.

Think of a supermanifold as a sheaf $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ of algebras locally modelled on $(\mathbb{R}^p, \mathcal{C}^{\infty p|q})$, where

$$\mathcal{C}^{\infty p|q}(U) = \mathcal{C}^{\infty}(U)[\xi_1, \ldots, \xi_q], \quad \xi_i \xi_j = -\xi_j \xi_i, U \subset \mathbb{R}^p.$$

The notions of a *super vector bundle* and a graded bundle generalize naturally to the notion of a *super graded bundle*, i.e., a *super fiber bundle* $\pi : \mathcal{E} \to \mathcal{M}$ in which one can distinguish a class of $\mathbb{N}$-graded fiber coordinates so that transition functions preserve this gradation We are interested in actions $h : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ of the monoid $(\mathbb{R}, \cdot)$ which we call a *homogeneity structure* on $\mathcal{M}$.

Example

Higher tangent bundles have their analogs in supergeometry. Given a supermanifold $\mathcal{M}$ a higher tangent bundle $\mathrm{T}^k\mathcal{M}$ is a natural example of a super graded bundle. For $k=2$ and local coordinates (x^A) on $\mathcal{M}$ (even or odd) one can introduce natural coordinates $(x^A,\dot{x}^B,\ddot{x}^C)$ on $\mathrm{T}^2\mathcal{M}$ where coordinates $\dot{x}^A$ and $\ddot{x}^A$ share the same parity as x^A and are of weight 1 and 2, respectively. Standard transformation rules apply:

$$x^{A'}=x^{A'}(x),\quad \dot{x}^{A'}=\dot{x}^B\frac{\partial x^{A'}}{\partial x^B},\quad \ddot{x}^{A'}=\ddot{x}^B\frac{\partial x^{A'}}{\partial x^B}+\dot{x}^C\dot{x}^B\frac{\partial^2 x^{A'}}{\partial x^B\partial x^C}.$$

A homogeneity structure h on a supermanifold $\mathcal{M}$ equips the body $|\mathcal{M}|$ with a (standard) *homogeneity structure*, and so $\underline{h_0} : |\mathcal{M}| \to |\mathcal{M}|_0$ is a (real) graded bundle over $|\mathcal{M}|_0 := h_0(|\mathcal{M}|)$. Every super graded bundle structure $\pi : \mathcal{E} \to \mathcal{M}$ provides $\mathcal{E}$ with a canonical homogeneity structure. The converse is also true:

Theorem

The categories of super graded bundles (with connected bodies) and homogeneity structures on supermanifolds (with connected bodies) are equivalent.

Sketch of proof. We need to show to show that given a homogeneity structure $h : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ on a supermanifold $\mathcal{M}$ one can always find an atlas with homogenous coordinates on $\mathcal{M}$. Without loss of generality we may assume that $|\mathcal{M}|_0$ is an open contractible subset $U \subset \mathbb{R}^n$, and $|\mathcal{M}| = U \times \mathbb{R}^{\mathrm{d}}$ is a trivial graded bundle over U, say of rank d, and $\mathcal{M} = \Pi E$ where $E = U \times \mathbb{R}^{\mathrm{d}} \times \mathbb{R}^q$ is a trivial vector bundle over $|\mathcal{M}| = U \times \mathbb{R}^{\mathrm{d}}$ with the typical fiber $\mathbb{R}^q$.

Denote coordinates (x^i, y^a, Y^A) on E, (x_i, y^a, ξ^A) on ΠE, so

$$h_t^*(x^i) = x^i + o(\mathcal{J}_{\mathcal{M}}^2),$$
$$h_t^*(y^a) = t^{\mathbf{w}(a)}\, y^a + o(\mathcal{J}_{\mathcal{M}}^2),$$
$$h_t^*(\xi^A) = \alpha_B^A(t, x^i, y^a)\xi^B + o(\mathcal{J}_{\mathcal{M}}^2), \qquad (*)$$

where $\mathcal{J}_{\mathcal{M}}$ is the maximal ideal in $\mathcal{O}_{\mathcal{M}}$. The crucial steps of the proof are the following

- h defines an action $\widetilde{h}$ on E which is given by

$$\widetilde{h}_t^*(x^i) = x^i, \quad \widetilde{h}_t^*(y^a) = t^{\mathbf{w}(a)}\, y^a, \quad \text{and} \quad \widetilde{h}_t^*(Y^A) = \alpha_B^A(t, x, y) Y^B.$$

- It is possible to replace Y^A with $\tilde{Y}^A = \gamma_B^A(x, y) Y^B$ so that $\widetilde{h}_t^*(\tilde{Y}^A) = t^{\mathbf{w}(A)}\, \tilde{Y}^A$. Then

$$h_t^*(\tilde{\xi}^A) = t^{\mathbf{w}(A)} \tilde{\xi}^A + o(\mathcal{J}_{\mathcal{M}}^2), \qquad (*')$$

where $\widetilde{\xi}^A := \gamma_B^A(x, y) \cdot \xi^B$. We complete the proof by showing:

Lemma

Consider a superdomain $\mathcal{M} = U \times \Pi\mathbb{R}^s$, $U \subset \mathbb{R}^r$, and introduce super coordinates $(y^1, \ldots, y^r, \xi^1, \ldots, \xi^s)$ on $\mathcal{M}$. Assume h is an action of $(\mathbb{R}, \cdot)$ on $\mathcal{M}$ such that

$$h_t^*(y^a) = t^{\mathbf{w}(a)}\, y^a + o(\mathcal{J}_{\mathcal{M}}^2), \quad \text{and} \quad h_t^*(\xi^i) = t^{\mathbf{w}(i)}\, \xi^i + o(\mathcal{J}_{\mathcal{M}}^2)\ .$$

Then

$$\left(\frac{1}{\mathbf{w}(a)!} \left. \frac{\mathrm{d}^{\mathbf{w}(a)}}{\mathrm{d}t^{\mathbf{w}(a)}} \right|_{t=0} h_t^*(y^a), \frac{1}{\mathbf{w}(i)!} \left. \frac{\mathrm{d}^{\mathbf{w}(i)}}{\mathrm{d}t^{\mathbf{w}(i)}} \right|_{t=0} h_t^*(\xi^i) \right)$$

are graded coordinates on the superdomain $\mathcal{M}$.

Thanks

Thank you for your attention!